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YALE UNIVERSITY

# LECTURES ON LIE GROUPS AND LIE ALGEBRAS

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# LECTURES ON LIE GROUPS AND LIE ALGEBRAS

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## LECTURE I

By a Lie group G we mean a set together with both a group structure and analytic structure satisfying the requirement that the mapping $(a,b) \longrightarrow ab^{-1}$ of $G \times G$ onto G should be analytic. We begin with a study of analytic manifolds.

## ANALYTIC STRUCTURES

Let M be a topological space. For each point $p \in M$, let there be given a family $A(p)$ whose elements consist of pairs $(U,f)$ where U is an open neighborhood of the point p and f is a real-valued function defined on U. We say that the assignment A of $A(p)$ to p defines an analytic structure on M if the following conditions are fulfilled :

A) For each $(U,f) \in A(p)$ and $q \in U$, we have $(U,f) \in A(q)$.

B) For each $p \in M$ there exists a set of elements $\{(U_i,f_i)\}$ $i = 1,2,\dots,r$ of $A(p)$ and an open set $V' \subseteq \bigcap_{i=1}^{r} U_i$ such that the mapping

$$q \longrightarrow (f_1(q), f_2(q),\dots,f_r(q))$$

is a homeomorphism of V' onto an open subset in real cartesian r-space $R^r$.

C) We further require that there exist for $\{(U_i, f_i)\}$ and for any $(U,f) \in A(q)$ with $q \in V'$ an absolutely convergent power series $P = \Sigma a_{i_1 \ldots i_r} (z_1 - f_1(q))^{i_1} \ldots (z_r - f_r(q))^{i_r}$ such that $f(x) = \Sigma a_{i_1 \ldots i_r} (f_1(x) - f_1(q))^{i_1} \ldots (f_r(x) - f_r(q))^{i_r}$ for all x in some neighborhood of q.

Definition

Any function f in A(p) is called analytic at p (with respect to A). We read condition C) as f depends analytically on $f_1, \ldots, f_r$ on V'. The set of functions $\{f_1, \ldots, f_r\}$ is called a system of coordinates on V' or also, a system of coordinates at p, or also a base for A(p).

D) We require in addition that if $\{f_1, \ldots, f_r\}$ is a system of coordinates on an open neighborhood V' and if f depends analytically on $f_1, \ldots, f_r$ on V', then $(U,f) \in A(p)$ for all p and U with $p \in U \subset V'$.

Let $\{f_1, \ldots, f_r\}$ be a system of coordinates on V' and let $\varepsilon$ be a positive real number. Set

$$V_\varepsilon = \{q ; | f_1(q) - f_1(p) | < \varepsilon , \ldots, | f_r(q) - f_r(p) | < \varepsilon\}$$

For all sufficiently small $\varepsilon$, $V_\varepsilon$ is mapped onto the cube of semi-edge $\varepsilon$ in cartesian space $R^r$. For such $\varepsilon$, $V_\varepsilon$ is

called the cubic neighborhood with center p and semi-edge $\varepsilon$ with respect to the coordinate system $\{f_1, \ldots, f_r\}$.

If we have an analytic structure A in a set M, we call the pair (M,A) an analytic manifold provided that M is a connected Hausdorff space. A function f defined on an open set U is called analytic on U with respect to the structure A if and only if $f \in A(p)$ for all $p \in U$.

The following is sometimes a convenient way of defining an analytic structure on a space M.

Let there be given a family of pairs $\{U^\alpha, \phi^\alpha)\}$ where :

1. $U^\alpha$ are open sets in M
2. $\bigcup_\alpha U_\alpha = M$
3. $\phi^\alpha$ is a homeomorphism of $U^\alpha$ with an open set of $R^r$.

Assume that for each pair $(\alpha, \beta)$ we have

$\phi^\beta \circ \phi^{\alpha^{-1}}$ is an analytic mapping of $\phi^\alpha (U^\alpha \bigcap U^\beta)$

We can then define on M an analytic structure by putting in A(p) all pairs (U,f) where U is an open neighborhood of p and f is a real-valued function on U satisfying :

$f \circ \phi^{\alpha^{-1}}$ is analytic on $\phi^\alpha(U^\alpha)$ , that is, it is expandable in a power series at each point of the open subset $\phi^\alpha(U^\alpha)$ of cartesian space.

If $p \in U^{\alpha}$ we can get a base for the analytic functions at p by letting $f_i(x) = i^{th}$ coordinate of $\emptyset^{\alpha}(x)$ for $x \in U^{\alpha}$.

## DIMENSION OF A MANIFOLD.

We note that if V is a cubic neighborhood of the point p with respect to $f_1,\ldots,f_r$ then V is homeomorphic to $R^r$. By the invariance of domain, p cannot have a neighborhood homeomorphic with $R^s$ unless $r = s$. Thus , if we say that $\dim_p M$ is r whenever p has a neighborhood homeomorphic to $R^r$ we have that $\dim_p M$ is an everywhere defined function of M into $Z^+$ ( the space of non-negative whole numbers and is continuous - since every point of a cubic neighborhood of p has the same dimension. Thus $\dim_p M$ is a constant when M is connected.

Definition

We say that an analytic manifold (M,A) has dimension r if every point has a neighborhood homeomorphic to $R^r$.

Note : One need not appeal to the topological definition of dimension to define the dimension of analytic manifolds.

If $(f_1,\ldots,f_r)$ and $(g_1,\ldots,g_s)$ are two sets of functions such that each is a system of coordinates on M at p, then by a consideration of Jacobians one can see that $r = s$. Hence the dimension of M could have been defined as the number of elements in a system of coordinates on M at any point.

## PRODUCT OF MANIFOLDS

Given two analytic manifolds $(M_1,A_1)$ and $(M_2,A_2)$ we define an analytic structure on the product space by taking $A(m_1,m_2)$ to consist of all functions depending analytically on $f_1,\ldots,f_r$, $g_1,\ldots,g_s$ where $f_1,\ldots,f_r$ is a base for $A_1(m_1)$ and $g_1,\ldots,g_s$ is a base for $A_2(m_2)$.

We will hardly have occasion to mention more than one structure on an analytic manifold. Accordingly, we write $M$ for $(M,A)$.

### Definition

A mapping $\emptyset$ of $M_1$ to $M_2$ is called analytic if for every function $f$ analytic on an open set $U$ in $M_2$, $f \circ \emptyset$ is analytic on $\emptyset^{-1}U$.

Note: The product structure that we introduced on $M_1 \times M_2$ is the smallest structure such that the projections onto $M_1$ and $M_2$ are analytic.

## PREVIEWS

We will follow the ideas of Sophus Lie in analyzing the structure of the Lie group G by means of certain "infinitesimal transformations". To suggest how this will be done, we consider the following :

- The group G operates on itself by means of left translation. That is, for each $a \in G$ the mapping $L_a: x \longrightarrow ax$ is an analytic mapping of G to G. In fact $L_a$ is a bianalytic homeomorphism of the underlying manifold, with itself,.

- The mapping $a \longrightarrow L_a$ is an (algebraic) isomorphism of G into the group of bianalytic homeomorphisms of G with itself.

- If U is any neighborhood of a topological group G and we let H be the subgroup generated by U, we have $hU \subset H$ for each $h \in H$ and hence H is open in G. Being an open subgroup, H is also closed.

Hence, for a connected Lie group G, each neighborhood U of the identity generates G and we write $G = \bigcup_n U^n$ where $U^n$ denotes the set of elements $a \in G$ which may be written $a = a_1 a_2 a_3 \ldots a_n$ with $a_i \in U$. Since $L_{ab} = L_a \circ L_b$, we may obtain the entire group of transformations $L_G$ by products of the transformations $L_U$ where U is an arbitrary neighborhood of the identity.

By taking infinitesimally small neighborhoods of the identity , we will, via a limiting process, recapture the operations of G on G by mere knowledge of the operation on G of the elements "infinitesimally near" the identity.

We will now clarify the notion of transformation "infinitesimally near" the identity by introducing the tangent space to the group G at the identity.

## LECTURE II.

### TANGENT SPACE TO A MANIFOLD AT A POINT.

If we think of a surface S embedded in euclidean 3-space, we have a clear picture of what is meant by a tangent vector to S at some point $p \in S$. It is a vector ( i.e. it has direction and magnitude) in 3-space. We show how it is possible to speak of tangent vectors even when we do not have an embedding.

Definition

Let p be a point on the manifold M. By a differentiable curve at p is meant a mapping $\alpha: I \longrightarrow M$ where $I = \{a \leq x \leq b\}$ with $\alpha(c) = p$ for some value $a \leq c \leq b$, and such that for any system of coordinates on M at p, say $\{f_1,\ldots,f_r\}$ we have :

$$\left.\frac{d\, f_i \circ \alpha}{d\, t}\right|_{t=c} \quad \text{exists for all } i = 1,2,\ldots,r.$$

Definition

Given two differentiable curves at p, $\alpha$ and $\psi: I \longrightarrow M$ with $\alpha(c) = \psi(c) = p$ we say $\alpha$ and $\psi$ are tangent at p in case

$$\left.\frac{d\, f_i \circ \alpha}{d\, t}\right|_{t=c} = \left.\frac{d\, f_i \circ \psi}{d\, t}\right|_{t=c}$$

An equivalence class of differentiable curves at p is called a tangent vector to M at p.

Note The chain rule for differentiation of functions of several variables shows that the above definitions do not depend on the system of coordinates $\{f_1, \ldots, f_r\}$.

Remark With the above definitions in mind we think of a tangent vector as telling how a curve "starts out" from the point p.

In the following we shall adapt another definition which is technically more efficient. It is based on the observation that if $\mathcal{U}$ and $\psi$ are tangent at $p \in M$ and f is analytic at p then

$$\frac{d}{dt}(f \circ \mathcal{U})\Big|_{t=c} = \frac{d}{dt}(f \circ \psi)\Big|_{t=c}$$

which follows easily from the fact that the equality holds for any system of coordinates. Thus, each tangent vector at p allows us to assign to any function f analytic at p a number - viz $\frac{d}{dt}(f \circ \mathcal{U})\Big|_{t=c}$ where $\mathcal{U}$ is a representative curve from the given tangent vector. For this reason we call a tangent vector at a point a "directional differentiation at p".

Definition

A tangent vector to a analytic manifold M at a point p is a mapping $X : A(p) \longrightarrow R$ (real numbers) satisfying

1) $X(f+g) = X(f) + X(g)$ for all $f, g \in A(p)$
2) $X(cf) = cX(f)$ for all $f \in A(p)$ and $c \in R$
3) $X(fg) = X(f)\, g(p) = f(p)\, X(g)$

**Note** Since $X(1) = X(1^2) = X(1)1 + 1X(1) = 2X(1)$ we have $X(1) = 0$ and hence $X(c) = 0$ for any constant c.

**Example**

Let $\{x^1, \ldots x^r\}$ be a system of coordinates at $p \in M$ and let $f \in A(p)$. We may regard f as a function on euclidean space $R^r$ by considering the mapping

$$\emptyset : q \longrightarrow (x^1(q), x^2(q), \ldots, x^r(q))$$

of a neighborhood of the point p onto a cube in $R^r$ - we have $f(q) = f \circ \emptyset^{-1}(x^1(q), \ldots, x^r(q))$. Then the directional derivative, $f \longrightarrow \left.\frac{\partial}{\partial x^1}\right|_p (f \circ \emptyset^{-1})$ is a tangent vector at p.

**Notation** Let $\{x^1, \ldots, x^r\}$ be a system of coordinates at p. Let $\emptyset$ be the mapping $q \longrightarrow \emptyset(q) = (x^1(q), \ldots, x^r(q))$ of a cubic neighborhood of p into the euclidean space $R^r$ (whose coordinates will be denoted by $\{u_1, \ldots, u_r\}$ ).

We define the symbol $\left.\frac{\partial}{\partial x^1}\right|_q (f)$ to mean $\left.\frac{\partial(f \circ \emptyset)}{\partial u^i}\right|_{\emptyset(q)}$

That we get the expected formula for directional differentiation is demonstrable as follows :

**Lemma 2.1**

Given i) $f \in A(p)$

ii) $\{x^1, \ldots, x^r\}$ a system of coordinates at p

iii) D a tangent vector at p

we have

$$Df = \left.\frac{\partial f}{\partial x^1}\right|_p Dx^1 + \left.\frac{\partial f}{\partial x^1}\right|_p Dx^2 + \ldots + \left.\frac{\partial f}{\partial x^r}\right|_p Dx^r$$

Proof

Setting $\phi(p) = (a^1,\ldots,a^r)$ and $A_i = \left.\dfrac{\partial(f\circ\phi^{-1})}{\partial u^i}\right|_{\phi(p)}$ we get

$$f\circ\phi^{-1} = f(p) + \sum_i A_i(u^i-a^i) + \sum_{i,j} A_{ij}(u^i-a^i)(u^j-a^j)$$

and hence $$f = f(p) + \sum_i A_i(x^i-a^i) + \sum_{i,j}(A_{ij}\circ\phi)(x^i-a^i)(x^j-a^j)$$

where the $A_{ij}$ are analytic functions around $\phi(p)$.

Taking D of both sides we get

$$Df = \sum_i A_i\, Dx^i = \sum_i \left.\frac{\partial f}{\partial x^i}\right|_p Dx^i$$

Definition

We set $\dot{M}_p = \{$ tangent vectors to M at p$\}$ and call this family of operators the tangent space to M at p.

Given $D_1$ and $D_2$ tangent vectors we define $(D_1+D_2)(f) = D_1f + D_2f$ and $(cD_1)f = c(D_1f)$ for any constant c. It is easily verified that $(D_1+D_2)$ and $(cD_1)$ are again tangent vectors and, so, the tangent space to M at p is a vector space over R.

Lemma 2.2

dim $\dot{M}_p$ = dim M. That is, the dimension of the tangent space to a point p is independent of the point p and is equal to the number of functions in a system of coordinates.

Proof.

The r tangent vectors $\frac{\partial}{\partial x^1}\Big|_p , \ldots , \frac{\partial}{\partial x^r}\Big|_p$ span $\dot{M}_p$ by the above lemma. Since

$$\frac{\partial x^i}{\partial x^j}\Big|_p = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$$

a relation $\sum_i c^i \frac{\partial}{\partial x^i} = 0$ implies $c^j = 0$ for $j = 1,2,\ldots,r$. Hence the vectors $\left\{\frac{\partial}{\partial x^i}\Big|_p\right\}$ are a basis for $\dot{M}_p$ and the result is proved.

An extremely important property of analytic mappings of $R^s$ into $R^t$ is that tangent curves are mapped to tangent curves. Because of the local nature of this statement we have the same for arbitrary manifolds.

Definition

Given an analytic mapping of the manifold M to the manifold N, $\mathcal{U}: M \longrightarrow N$, the induced mapping

$$d\mathcal{U}_p : \dot{M}_p \longrightarrow N_{\mathcal{U}(p)} \text{ defined by}$$

$$d\mathcal{U}_p(X)f = X(f \circ \mathcal{U}) \text{ for any } X \in \dot{M}_p \text{ and } f \in A(\mathcal{U}(p))$$

is called the differential of $\mathcal{U}$ at p.

The verification that $Y = d\mathcal{U}_p(X)$ is a tangent vector is straightforward and it is easily seen that $d\mathcal{U}_p$ is a linear mapping of $\dot{M}_p$ into $\dot{N}_{\mathcal{U}(p)}$.

Example

Let $M = R$ and $\mathcal{U}:R \longrightarrow N$ an analytic path. Then

$$d\mathcal{U}_p\left(\frac{d}{dt}\Big|_p\right) = \frac{d}{dt}\Big|_{t=p}[(f \circ \mathcal{U})t]$$

Definition

Given analytic manifolds $M$ and $N$ and an analytic mapping $\mathcal{U}: M \longrightarrow N$ carrying $p \in M$ into $q = \mathcal{U}(p) \in N$ and systems of coordinates $\{x^1,\dots,x^r\}$ at $p$ and $\{y^1,\dots,y^s\}$ at $q$ we form the matrix of $d\mathcal{U}_p$ with respect to the bases

$$\left\{\frac{\partial}{\partial x^1}\Big|_p, \frac{\partial}{\partial x^2}\Big|_p, \dots, \frac{\partial}{\partial x^r}\Big|_p\right\} \text{ for } \dot{M}_p$$

$$\text{and } \left\{\frac{\partial}{\partial y^1}\Big|_q, \frac{\partial}{\partial y^2}\Big|_q, \dots, \frac{\partial}{\partial y^s}\Big|_q\right\} \text{ for } \dot{N}_q$$

This matrix is an $r\times s$ matrix with constant coefficients whose entries are given by

$$c_{ij} = \frac{\partial(y^j \circ \mathcal{U})}{\partial x^i}\Bigg|_p \qquad \begin{matrix} i = 1,\dots,r \\ j = 1,\dots,s \end{matrix} \qquad ;$$

it is called the functional matrix.

**Theorem 2.3**

If $d\mathcal{U}_p$ is a monomorphism, then from among the s functions $y^1 \circ \mathcal{U}, y^2 \circ \mathcal{U}, \ldots, y^s \circ \mathcal{U}$ it is possible to select a system of coordinates at p.

**Proof**

We can select $y^{i_1} \circ \mathcal{U}, \ldots, y^{i_r} \circ \mathcal{U}$ so that

$$\det \left( \frac{\partial y^{i_j} \circ \mathcal{U}}{\partial x^i} \right) \neq 0.$$

Set $z^j = y^{i_j} \circ \mathcal{U}$

Set $\begin{cases} Z : q \longrightarrow (z^1(q), \ldots, z^r(q)) \\ X : q \longrightarrow (x^1(q), \ldots, x^r(q)) \end{cases}$

The determinant above is the Jacobian of $Z \circ X^{-1}$ and we may use the implicit function theorem to conclude that $Z \circ X^{-1}$ has an analytic inverse. Then $Z^{-1} = X^{-1} \circ (Z \circ X^{-1})^{-1}$ exists and $X \circ Z^{-1} = (Z \circ X)^{-1}$. It follows that $X \circ Z^{-1}$ is analytic and thus $x^1, \ldots, x^r$ depend analytically on $z^1, \ldots, z^r$ around p. Hence $\{z^1, \ldots, z^r\} = \{y^{i_1} \circ \mathcal{U}, \ldots, y^{i_r} \circ \mathcal{U}\}$ is a system of coordinates at p.

**Theorem 2.4**

If $d\mathcal{U}_p$ is an epimorphism, then $y^1 \circ \mathcal{U}, \ldots, y^s \circ \mathcal{U}$ can be enlarged to a coordinate system.

Proof.

The functional matrix $\left(\frac{\partial y^{i} \circ \mathcal{C}}{\partial x^{1}}\right)$ is an rxs matrix whose rank is s.

Now, since the rank is s, after suitably reordering the $\{x^{j}\}_{j=1,\dots,r}$ and setting $z^{i} = \begin{cases} y^{i} \circ \mathcal{C} & \text{for } i=1,\dots,s \\ x^{i} & \text{for } i = s+1,\dots,r \end{cases}$

we get the functional matrix

$$\left(\frac{\partial z^{i}}{\partial x_{j}}\Big|_{p}\right) = \left(\begin{array}{c|c} \frac{\partial y^{i} \circ \mathcal{C}}{x^{i}}\Big|_{p} & 0 \\ \hline * & E_{r-s,r-s} \end{array}\right)$$

[ in which $E_{r-s,r-s}$ is the unit matrix of rank r-s]

which is non-singular. Hence $y^{1} \circ \mathcal{C}$, $y^{2} \circ \mathcal{C},\dots,y^{s}\ \mathcal{C}$, $x^{s+1},\dots,x^{r}$ is a system of coorinates at p.

Theorem 2.5

If $d\mathcal{C}_{p}$ is an isomorphism, then $\mathcal{C}^{-1}$ is analytic at $\mathcal{C}(p)$

Proof.

It results immediately from theorems 2.3 and 2.4 that the functions $y^{1} \circ \mathcal{C},\dots, y^{s} \circ \mathcal{C}$ are a system of coordinates at p. Theorem 2.5 is then a consequence of the fact that $y^{i} \circ \mathcal{C} \circ \mathcal{C}^{-1} = y^{i}$ for $i = 1,\dots,s$ are analytic by hypothesis.

Theorem 2.6

$$d\mathcal{U}_p = 0 \text{ for all } p \Longrightarrow \mathcal{U} \text{ is constant.}$$

Proof.

It suffices to show that $\mathcal{U}$ is constant in some neighborhood of p ( for p fixed).

Let $\{x^1, \ldots, x^r\}$ be a system of coordinates at p and let $\{y^1, \ldots, y^s\}$ be a system of coordinates at $q = \mathcal{U}(p)$.

By hypothesis we have

$$\frac{\partial y^j \circ \mathcal{U}}{\partial x^i} = 0 \text{ for each } i \text{ and } j.$$

Therefore $y^j \circ \mathcal{U}$ is constant in a neighborhood of p. That is to say $y^i(\emptyset\,(q)) = y^i(\emptyset(p))$ $\quad i = 1,2,\ldots,s$. As the $\{y^i\}$ are a system of coordinates we must have $\emptyset\,(q) = \emptyset\,(p)$ is a neighborhood of p.

Remark.

In the special case that M = V, a real vector space is made into a manifold in the obvious way; viz. we take as analytic functions at p all these functions which depend analytically around p on linear functions. We identify $\dot{V}_p$ with V in the following manner :

Let $\lambda$ be any linear function on V and X any element of $\dot{M}_p$. As $\lambda$ is analytic the elements of $\dot{M}_p$ are contained in $V^{**}$ (i.e. the dual space of $V^*$, which is itself the dual space of V). We have a natural isomorphism $V^{**} \approx V$ given by

$$V^{**} \ni X \longleftrightarrow \overset{\vee}{X} \in V \text{ where}$$

$$X(\lambda) = \lambda(\overset{\vee}{X}) \quad \text{for all } \lambda \in V^*$$

In particular if N is an analytic manifold and $\varphi : N \longrightarrow V$ is analytic we can identify the differential $d\varphi_p : \dot{N}_p \longrightarrow \dot{V}_{\varphi(p)}$ with a linear mapping of $\dot{N}_p \longrightarrow V$.

As a special case, viz. V = R, the differential $d\varphi$ can be regarded as a real-valued function on $\tau(N)$ (The tangent bundle of N which is merely the disjoint union of $\dot{N}_p$ for all p, as a point-set) which is linear on $\dot{N}_p$ for each p.

Note

If M is an analytic manifold and U an open subset of M, we may regard U as an analytic manifold in a natural way. Specifically, a function f is said to be analytic at $p \in U$ if it is the restriction to U of a function analytic at p on M. This is the same thing as saying that the identity map $i : U \longrightarrow M$ is analytic. In this case the differential $di_p$ is an isomorphism for each $p \in U$.

Note

We point out here that the chain differentiation rule for analytic manifolds can be easily verified from the definition of the differential.

Theorem

If $M \xrightarrow{\varphi} N \xrightarrow{\psi} p$ is a sequence of analytic mappings then $d(\psi \circ \varphi) = d\psi \circ d\varphi$.

## LECTURE III.

### ANALYTIC VECTOR FIELDS ON M

By a vector field X on a manifold M we mean a function which assigns to each point $p \in M$ a tangent vector $X(p)$ in $\dot{M}_p$. The vector field X is called an analytic vector field if for each function f analytic around p the function $Xf : p \longrightarrow X(p)f$ is analytic around p. In terms of a coordinate system $\{x^1,\dots,x^r\}$ at p an arbitrary vector field may be written:

$$X(q) = X(q)x^1 \left.\frac{\partial}{\partial x^1}\right|_q + X(q)x^2 \left.\frac{\partial}{\partial x^2}\right|_q + \dots + X(q)x^r \left.\frac{\partial}{\partial x^r}\right|_q$$

We see that the above definition is equivalent to :

The vector field X is analytic if for any system of coordinates $\{x^1,\dots,x^r\}$ on any neighborhood V, the functions $q \longrightarrow X(q)x^i$ $(i;1,\dots r)$ are analytic on V.

Thus an analytic vector field is a linear combination of the vector fields $\frac{\partial}{\partial x^i}$ whose coefficients are analytic.

### INFINITESIMAL TRANSFORMATIONS

We will now see how an analytic vector field gives rise to an "infinitesimal transformation" on M.

Let $X$ be an analytic vector field on $M$. Let $I_{a,b}$ be the open interval $a < t < b$.

Definition

An analytic mapping $\mathcal{C}: I \longrightarrow M$ satisfying the condition

$$d\mathcal{C}\,\frac{d}{dt}\Big|_x = X(\mathcal{C}(x)) \quad \text{for each } x \text{ in } I.$$

is called a trajectory to X on I. If 0 is in $I$ we call the point $\mathcal{C}(0) \in M$ the origin of the trajectory .

We establish the existence and uniqueness of trajectories in the following theorem.

Theorem 3.1

Let $X$ be an analytic vector field on $M$ and $p \in M$. Then there exist

i) A neighborhood $U$ of $p$

ii) A positive real number $\epsilon$

iii) An analytic mapping $\mathcal{C}: U \times I_{-\epsilon,\epsilon} \longrightarrow M$ such that for each $q \in U$, the mapping

$$\mathcal{C}(q,t) : I_{-\epsilon,\epsilon} \longrightarrow M$$

is the unique trajectory to $X$ with origin $q$ on the interval $I_{-\epsilon,\epsilon}$.

Proof.

The condition required is that for any function f analytic around a point $\mathcal{Q}(p,t)$ we have :

1) $\frac{df}{dt}(\mathcal{Q}(q,t)) = (Xf)(\mathcal{Q}(q,t))$ for all $q \in V$ and $|t| < \epsilon$

2) $\mathcal{Q}(q,0) = q$

Let $\{x^1,\ldots,x^r\}$ be a system of coordinates at p on a neighborhood V. In order to satisfy condition 1), it suffices that 1) $\frac{dx^i}{dt}(\mathcal{Q}(q,t)) = (Xx^i)(\mathcal{Q}(q,t))$ for $i = 1,2,\ldots,r$.

Let $X : q \longrightarrow (x^1(q),\ldots,x^r(q))$ be the mapping of $V \longrightarrow R^r$. In terms of the coordinates on the neigborhood X(V) in euclidean space

1') becomes after setting

$u^i = x^i \circ X^{-1}$

$a^i = x^i(p)$

$A^i = Xx^i$

$B^i = A^i \circ X^{-1}$

$\psi(q,t) = X(\mathcal{Q}(q,t))$

1′) $\frac{dx^i \circ X^{-1}}{dt}(\psi(q,t)) = A_i \circ X^{-1} \quad (\psi(q,t))$

or

1″) $\frac{du^i}{dt}(\psi(q,t)) = B_i(\psi(q,t))$ with $\psi(q,0) = X(q)$

The fundamental existence and uniqueness theorem on ordinary differential equations guarantees the existence of a positive real number $\epsilon$ and an analytic mapping $\xi : (u^1,\ldots,u^r;t) \longrightarrow V$

for $\begin{cases} |u^i-a^i| < \epsilon \\ |t| < \epsilon \end{cases}$ such that

1) $\frac{du^i}{dt}\ (\xi(u;t)) = B_i(\xi(u;t))$, where $u = (u^1,\ldots,u^r)$

2) $\xi(u;0) = u$

Setting $\psi(q,t) = \xi(X^{-1}\ u;\ t)$ and $\varphi(q,t) = X^{-1}\ \psi(q,t)$ we have the desired mapping $\varphi\colon V\times I_{-\epsilon,\epsilon} \longrightarrow M$.

Lemma 3.2

If $\varphi$ is a trajectory to X on an interval $I_{a,b}$ and c is any real number, then $t \longrightarrow \varphi(t+c)$ is a trajectory to X on the interval $I_{a-c,b-c}$

Proof

The mapping $L_c\colon t \longrightarrow t+c$ of $I_{a-c,b-c}$ to $I_{a,b}$ sends $\frac{d}{dt}\Big|_x$ to $\frac{d}{dt}\Big|_{x+c}$ since $\frac{d}{dt}\, f(t+c) = f'(t+c)$ for f analytic. Setting $\psi(t) = \varphi(t+c) = \varphi \circ L_c(t)$ we get

$$d\psi\left(\frac{d}{dt}\Big|_x\right) = d\varphi \circ dL_c\left(\frac{d}{dt}\Big|_x\right) = d\varphi\left(\frac{d}{dt}\Big|_{x+c}\right) = X(\varphi(t+c))$$
$$= X(\psi(t))$$

Corollary to Theorem 3.1

If $\varphi$ and $\psi$ are trajectories to X on $I_{a,b}$ with $\varphi(c) = \psi(c)$ for some c in $I_{a,b}$ then $\varphi$ and $\psi$ coincide throughout $I_{a,b}$.

Proof.

Using the Lemma and theorem 3.1 we see that if $\varphi$ and $\psi$ coincide at a point of $I_{a,b}$, they coincide in some neighborhood of that point. Thus, the set of points on which $\varphi$ and $\psi$ coincide is open. This subset is clearly closed. Hence $\varphi$ and $\psi$ coincide on the entire interval.

We know from ordinary differential equations that we cannot expect to be able to define the trajectories for all $t \in R$. Theorem 3.2. states that we may speak of a trajectory with largest domain of definition

**Definition**

Let I and $I'$ be open intervals with $I' \subset I$. Let $\mathcal{C}$ and $\mathcal{C}'$ be trajectories to X on I and $I'$ respectively and let $\mathcal{C}$ restricted to I coincide with $\mathcal{C}'$. Then we say $\mathcal{C}' < \mathcal{C}$.

**Theorem 3.2**

Let X be an analytic vector field on a manifold M and let p be any point of M. There exists a unique maximal trajectory to X with origin p.

**Proof**

We remark that we here view a trajectory as a subset of $M \times R$. The subsets are partially ordered by inclusion and this is the same as the ordering defined above in view of corollary to theorem 3.1.

Let T be the union of all trajectories to X with origin p. T is a well-defined function because of the corollary to theorem 3.1. It is a trajectory and $T_1 \leq T$ for any trajectory $T_1$ with origin p.

EXPONENTIAL

Definition

Let $t \longrightarrow \mathcal{U}(p,t)$ denote the maximal trajectory to X with origin p. $\mathcal{U}(p,t) : I_{a,b} \longrightarrow M$ and $\mathcal{U}(p,0) = p$. Set $D = \{(p,t) ; \mathcal{U}(p,t)$ is defined $\}$ and $D_{t_0} = \{p ; (p,t_0) \in D\}$ The set D, and hence $D_{t_0}$, is open by virtue of theorem 3.1

Definition

The mapping $p \longrightarrow \mathcal{U}(p,t)$ for $p \in D_t$ is called exp tX(p). It is an analytic mapping with domain $D_t$ for a fixed t.

# LECTURE IV

## EXPONENTIAL CONTINUED

**Theorem 4.1**

$\exp\,(t+s)\,X\,(p) = \exp tx \circ \exp sX(p)$ whenever the right side is defined.

**Proof.**

If $\alpha$ is a trajectory to X with origin p then $t \longrightarrow \alpha(t+c)$ is a trajectory to X with origin $\alpha(c)$ (Lemma 3.2). By the uniqueness of trajectories (Corollary 3.3) we know that $t \longrightarrow \exp\,(t+s)X(p)$ coincides with the trajectory $t \longrightarrow \exp tX(\exp sX(p))$ for all t such that $(\exp sX(p), t)$ is in the set D.

**Exercises**

1. Let M be a compact analytic manifold and X an analytic vector field.
    a) Show $D = M \times R$
    b) The mapping exp tX is a bi-analytic mapping of M to M
    c) The map $t \longrightarrow \exp tX$ is a homomorphism of the additive group of real numbers to the group of bianalytic maps of M
2. A point $p \in M$ is fixed under exp tX for all $t \Longleftrightarrow X(p) = 0$
3. Let $M = R^1$. Find D for
    a) $X = x \dfrac{d}{dx}$
    b) $X = x^2 \dfrac{d}{dx}$

Definition

Let $\mathcal{U} : M \longrightarrow N$ be an analytic mapping of the analytic manifolds M and N. Let f be a function analytic around $\mathcal{U}(p)$ for some $p \in M$. Then $f \circ \mathcal{U}$ is analytic around p and we call the map $\tilde{\mathcal{U}} : f \longrightarrow f \circ \mathcal{U}$ the transpose of $\mathcal{U}$.

The following theorem justifies the use of the terminology "exp" in this context.

Theorem 4.2

Let f be a function analytic around $p \in M$. There exists a positive real number a such that

$$f(\exp tX(p) = \sum_{n=0}^{\infty} \frac{t^n}{n!} X^n f(p) \text{ for } |t| < a.$$

Moreover this equation holds whenever the right side exists.

Proof.

$\frac{d}{dt} g(\exp tX(p)) = Xg(\exp tX(p))$ for any analytic function g. Hence $\frac{d^n}{dt^n}\Big|_{t=0} f(\exp tX(p)) = X^n f(p)$. Thus we see that the right-hand side is just the Taylor series for $f \circ \exp tX(p)$ and the theorem is proved.

The formula of theorem 4.2 has the following sheaf-theoretic interpretation :

We view the manifold M as a sheaf A over the underlying topological space with stalk A(p) at p. The vector field X can be regarded as an operator taking A(p) into itself. The operation

$\widetilde{\exp}\, X : f \longrightarrow f \circ \exp X$ is defined on a subsheaf A' and maps $A'(p)$ into $A(p)$. The operator $f \longrightarrow \sum_{n=0}^{\infty} \frac{X^n f}{n!}$ is defined in a subsheaf $A''$ and maps $A''(p) \longrightarrow A(p)$. Then theorem 4.2 states that $A' \supset A''$ and that $\widetilde{\exp}\, X$ is an extension of $\sum_{n=0}^{\infty} \frac{X^n}{n!}$ i.e. $\widetilde{\exp}\, X = \sum_{n=0}^{\infty} \frac{X^n}{n!}$ on $A''$

Note

The term "infinitesimal transformations" is used in the literature to mean the analytic vector field X. The transformations exp tX are called "finite transformations"

SUMMARY

Given an analytic vector field X on manifold M we form the family G of (so-called "finite") transformations exp tX. This family has the property that there is a well-determined open subset $D \subset R \times M$ such that $\exp t_0 X(p)$ is defined and analytic on the open set $D_{t_0} = \{(t_0 p); (t_0, p) \in D\}$. For fixed $p_0 \in M$ $\exp tX(p_0) : I_{a,b} \longrightarrow M$ is the maximal trajectory to X on M with origin $p_0$.

ANALYTIC FAMILIES OF TRANSFORMATIONS

Given the analytic vector field X on M, the associated family G has a "group-like" character. More specifically, each transformation exp tX in G has an inverse, viz exp -tX. Furthermore, the family G has a property resembling closure under composition since $\exp t_1 X \circ \exp t_2 X = \exp(t_1 + t_2) X \in G$ whenever the left-side is defined ( theorem 4.1)

Thus, abstractly, G is a set of homeomorphisms mapping open sets of M into open sets of M and satisfies :

1) for each T in G, $T^{-1}$ is in G.

2) for each pair, $T_1, T_2$ in G with domain $T_2 \cap$ image $T_1 \neq \emptyset$ we have $T_2 \circ T_1$ is the restriction to $T_1^{-1}$ (domain $T_2 \cap$ image $T_1$) of an element of G.

We introduce the following general notion in order to formulate the connection between a Lie group and its infinitesimal transformations.

## Definition of analytic family of transformations.

Let P and M be analytic manifolds and let D be an open set in $P \times M$ Let $\mathcal{U}: D \longrightarrow M$ be an analytic mapping and for each $g \in P$, let $\mathcal{U}_g$ be defined by $\mathcal{U}_g(x) = \mathcal{U}(g,x)$ for (g,x) contained in D. Denote by $D_g$ the set of all $x \in M$ such that (g,x) is in D. If $\mathcal{U}_g$ is a bianalytic map of $D_g$ onto $\mathcal{U}_g(D_g)$ for each $g \in P$, we call the triple $(P, M, \mathcal{U})$ an analytic family of transformations in M. P is called the parameter space of the family.

Let E be a non-empty open set in M and $E \subset D_a \cap D_b$ for some a and b in P with $\mathcal{U}_a = \mathcal{U}_b$ on E. Then we write $a = b(E)$ and this is an equivalance relation in P. If U and V are subsets of P we write $U \subset V(E)$ if for every $a \in U$ there exists a $b \in V$ with $a = b(E)$. If a,b,c are elements of P such that $\mathcal{U}_b \circ \mathcal{U}_a$ and $\mathcal{U}_c$ are each defined on E and coincide on E, we write $b \circ a = c(E)$. If 1 is an element of P

with $D_1 = M$ and $\varphi_1$ = the identity map of M onto itself, we call 1 an identity of P. Let 1 be an identity of P and let K be an arbitrary compact subset of M; then we can find a neighborhood U of 1 such that $U \times K \subset D$; that is, $K \subset D_g$ for all $g \in U$.

Definition of pseudogroup

An analytic family of transformations $(P, M, \varphi)$ is called a pseudogroup if the following conditions are satisfied :

a) P contains an identity , 1.

b) For any relatively compact domain E in M (i.e. $\bar{E}$ is compact) we have :

1) For each neighborhood U of 1, there is a neighborhood V of 1 and an analytic mapping $g \longrightarrow g'$ of $V \longrightarrow U$ such that $E \subset D_g \cap D_{g'}$ and $\varphi_{g'} = \varphi^{-1}_g$ on E

NOTE : As we shall see in Lecture VI this condition is actually a consequence of the others.

2) Given $g \in P$ with $D_g \supset E$ and given a neighborhood U of g, there exist neighborhoods V of 1 and W of g and an analytic mapping $\mu\ V \times W \longrightarrow U$ such that

i) $\mu(h,k) = h \circ k$ (E)

ii) $h \longrightarrow \mu(h,g)$ is regular at 1.

# THE LOGARITHMIC DERIVATIVES OF AN ANALYTIC FAMILY

Let $(P, M, \mathcal{U})$ be an analytic family of transformations in M. For each tangent vector X to P at $g \in P$ (i.e. for $X \in \dot{P}_g$) we shall associate an infinitesimal transformation on the domain $D_g$ as follows : let $\psi$ denote the mapping

$$\psi : h \longrightarrow \mathcal{U}_h \circ \mathcal{U}^{-1}{}_g(x)$$

which is defined on a neighborhood of g in P. We define $\lambda(X)_x = d\psi_h(X)$ for $x \in D_g$.

An alternative description of $\lambda(X)$ can be given in the following way. Let $g(t)$ be a differentiable path in P with $g(t_0) = g$, and assume that the tangent at $t_0$ is X. Thus for any function f analytic around $x \in M$ we have :

$$\lambda(X)_x f = \frac{d}{dt}\Big|_{t=t_0} f\Big(\mathcal{U}_{g(t)}\, \mathcal{U}^{-1}{}_{g(t_0)} \quad (x)\Big)$$

In other words :

$$\lambda(X) = \frac{d}{dt}\Big|_{t=t_0} \quad \tilde{\mathcal{U}}^{-1}{}_{g(t_0)} \quad \tilde{\mathcal{U}}_{g(t)}$$

where $\tilde{\mathcal{U}}$ is the transpose of the mapping $\mathcal{U}$. The infinitesimal transformation $\lambda(X)$ is called _the logarithmic derivative of the family along the tangent vector X_ . If $X \in \dot{P}_g$, $\lambda(X)$ is defined on $D_g$. If, in particular, $X \in \dot{P}_1$, $\lambda(X)$ is defined throughout M. We set $\mathcal{L}_g = \lambda(\dot{P}_g)$ and we say _$\mathcal{L}_g$ is independent of g_ if $\mathcal{L}_g$ coincides with $\mathcal{L}_1$ on $D_g$.

# LECTURE V

We now state the fundamental theorems of Sophus Lie. Theorem 5.1 will be proven in lecture VI and Theorem 5.2 in lecture VII. A proof of theorem 5.3 will be given later.

## THE FUNDAMENTAL THEOREMS OF LIE

### Theorem 5.1

Let $(P, M, \mathcal{U})$ be an analytic family of transformations in M having an identity 1. Assume that on $\dot{P}_1$, logarithmic differentiation $X \longrightarrow \lambda(X)$ is a one-one mapping. Then $(P, M, \mathcal{U})$ is a pseudogroup if and only if $\mathcal{L}_g = \lambda(\dot{P}_g)$ is independent of the particular choice of g in P,

Moreover there is a neighborhood U of 0 in the linear space $\mathcal{L}$, an open non-empty set E in M, and an analytic map $\rho : U \longrightarrow P$ such that

1) $\rho(0) = 1$

2) $d\rho_0(\lambda(X)) = X$ for all $X \in \dot{P}_1$ (We have identified $\dot{\mathcal{L}}_0$ with $\mathcal{L}$ here as pointed out in lecture II.)

3) exp X coincides with $\mathcal{U}_{\rho(g)}$ on E for all $g \in U$..

Given infinitesimal transformations X and Y on the manifold M, we readily verify that XY - YX is again an infinitesimal transformation : it is certainly linear and we have :

$$\cancel{(XY-YX)(fg)} = X(Yf\cdot g + f\cdot Yg) - Y(Xf\cdot g + f\cdot Xg) =$$

$$(XY-YX)f\cdot g + f\cdot(XY-YX)g$$

We denote XY - YX by the symbol $[X,Y]$ and verify easily that

1) $[\alpha_1 X_1 + \alpha_2 X_2, Y] = \alpha_1[X_1,Y] + \alpha_2[X_2,Y]$ (Linearity)

2) $[X_1,Y] = -[Y,X]$ (Anti-symmetry)

3) $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$ (Jacobi identity)

Definition

A finite-dimensional vector space $\mathcal{L}$ together with an operation $[\ ,\ ] : \mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{L}$ satisfy8ng the three properties above is called a Lie algebra.

We define ad $X : Y \longrightarrow [X,Y]$ and note that the mapping ad : $X \longrightarrow$ ad $X$ is a homomorphism of the Lie algebra $\mathcal{L}$ into the Lie algebra Hom $(\mathcal{L}, \mathcal{L})$, the latter made into a Lie algebra by taking $[A,B] = AB-BA$ for $A,B \in \text{Hom}(\mathcal{L}, \mathcal{L})$). That is

$$\text{ad } [X,Y] = [\text{ad } X, \text{ad } Y]$$

Moreover, for any X in $\mathcal{L}$, ad X is a derivation of $\mathcal{L}$, that is

$$\text{ad } X([Y,Z]) = [\text{ad } X(Y), Z] + [Y, \text{ad } X(Z)].$$

Theorem 5.2

Let $\mathcal{L}$ be a linear family of infinitesimal transformations on a manifold M. Let $L = \{\exp X;\ X \in \mathcal{L}\}$. Then if L is a pseudogroup of transformations in M we have $[X,Y] \in \mathcal{L}$ for any X,Y in $\mathcal{L}$.

Conversely; if $[X,Y] \in \mathcal{L}$ whenever $X,Y \in \mathcal{L}$, then there exists a neighborhood U of 0 in $\mathcal{L}$ such that exp U is a pseudogroup of transformations in M.

The linear family $\mathcal{L}$ is called the Lie algebra of the pseudogroup L.

## Theorem 5.3

Given a Lie algebra A over the field of real numbers, there exists a manifold M and a Lie algebra of infinitesimal transformations on M that is isomorphic to A.

## Exercises

1. Let $M = R^1$. Let $\mathcal{L}$ be the linear family of infinitesimal transformations generated by $X = \frac{d}{dx}$ and $Y = x\frac{d}{dx}$. We readily verify that $[\mathcal{L}, \mathcal{L}] \subset \mathcal{L}$ and in this case $L = \{\exp X;\ X \in \mathcal{L}\}$ is actually a group.

2. Let $M = R^1$ and $X = \frac{d}{dx}$, $Y = x\frac{d}{dx}$, $Z = x^2\frac{d}{dx}$. The linear span $\mathcal{L}$ of X, Y and Z is again a Lie algebra and L is the group of projective transformations on the line.

3. Let G be the group of rigid notions of the plane. Find the associated family of infinitesimal transformations. Do the same for the group of rigid notions of the 2-sphere $S^2$.

# LECTURE VI

We begin by observing that if $(P, M, \mathcal{U})$ is a pseudogroup, then $\mathcal{L}_g = \lambda(\dot{P}_g)$ is independent of the choice of g, in the sense of lecture IV. To see this, let $c(t)$ be a differentiable path in P with $c(0) = a$. Since right-translation by a is defined in a neighborhood of 1 and is regular at 1 we may write $c(t) = c'(t) \circ a$ where $c'(t)$ is a differentiable path satisfying $c'(0) = 1$. We then have $c(\tilde{t}) = \tilde{a} \circ c'(\tilde{t})$ and

$$\frac{d}{dt}\ c(\tilde{t}_0)^{-1}[c(\tilde{t})] = \frac{d}{dt}\ [\widetilde{c'(t_0)}^{-1}\ \tilde{a}^{-1}][\tilde{a}\ \widetilde{c'(t)}\ ]$$

$$= \frac{d}{d(t)}\ \widetilde{c'(t_0)}^{-1}\ [\widetilde{c'(t)}]$$

showing that $\mathcal{L}_a = \mathcal{L}_1$.

We now prove the converse. Namely, we assume that $\mathcal{L} = \mathcal{L}_g = \lambda(\dot{P}_g)$ is independent of g and that the analytic family $(P, M, \mathcal{U})$ contains an identity, 1. Also assume that logarithmic differentiation $X \longrightarrow \lambda(X)$ is a one-to-one mapping. We will show that $(P, M, \mathcal{U})$ is a pseudogroup.

Let c be a point of P. As $\mathcal{L} = \mathcal{L}_c$ for all c, the logarithmic differentiation $\lambda$ on $\dot{P}_c$ gives a map $\Lambda$ of the tangent bundle $\mathcal{T}(P)$ of P onto $\mathcal{L}$ - a bundle over a point. $\Lambda$ is an analytic fiber mapping. More explicitly - let U be a coordinate neighborhood in P. Then the tangent bundle $\mathcal{T}(U)$ is identified with $U \times R^n$ where $n = \dim P$. Define $\Lambda_\alpha : U \longrightarrow \mathcal{L}$ by $\Lambda_\alpha(c) = \Lambda(c, \alpha) = \alpha^*$ for $c \in U$ and $\alpha$ a fixed vector in $R^n$. Then we have :

Lemma:

$\Lambda_{\alpha}(c)$ is analytic in c.

Proof

An arbitrary linear function $\xi$ on $\mathcal{L}$ is of the form $X \longrightarrow Xf(p)$ where f is analytic at p. We must check that $(\xi \circ \Lambda)(c,\alpha)$ is analytic in c. This can be rewritten :

$$(\xi \circ \Lambda)(c,\alpha) = \alpha^{*}_{c} f(p) = \frac{d}{dt}\Big|_{t=0} f(\mathcal{C}_{c+t\alpha} \circ \mathcal{C}_{c}^{-1}(p))$$

which is analytic in c since $\mathcal{C}$ is analytic in $c \in P$ and $p \in M$.

For notational convenience, we denote $\mathcal{C}_c$ by c.

Let $U'$ be a neighborhood of $a \in P$. Fix a coordinate neighborhood $V'$ of $1 \in P$ and let b(t) be an analytic path in $V'$ with $b(0) = 1$. Denote the coordinates of b(t) by $(\beta_1(t),\ldots,\beta_n(t)) = \beta(t)$. We wish to prove that there is a path c(t) with

① $\quad b(t) \circ a = c(t) \qquad$ (E)

whenever b(t) lies in a suitable neighborhood $V \subset V'$. We derive a differential equation for c(t) upon taking logarithmic derivatives of both sides of ① - viz :

$$\widetilde{c(t)} = \tilde{a} \circ \widetilde{b(t)} \quad \text{and}$$

$$\frac{d}{dt}\Big|_{t=t_0} \tilde{c}(t_0)^{-1}[c(\tilde{t})] = \frac{d}{dt}\Big|_{t=t_0} [b(\tilde{t}_0)\ \tilde{a}^{-1}][\tilde{a}\ b(\tilde{t})]$$

$$= \frac{d}{dt}\Big|_{t=t_0} b(\tilde{t}_0)^{-1}\ [b(\tilde{t})]$$

that is

$$② \qquad \lambda_{c(t)}\ (\dot{c}(t)) = \lambda_{b(t)}\ (\dot{b}(t)) \qquad \text{for each } t.$$

From ② and the Lemma above there results the differential equation

$$\textcircled{I} \qquad \dot{c}(t) = F(c(t),\ b(t))\ [\dot{b}(t)] = G(c,t)\dot{b}$$

where $\dot{b}(t)$ and $\dot{c}(t)$ (the tangent vectors to $b(t)$ and $c(t)$, respectively) are in $\dot{P}_b$ and $\dot{P}_c$, respectively, and $F$ is the linear mapping $\lambda_c^{-1} \circ \lambda_b$ of $\dot{P}_b$ onto $\dot{P}_c$. Equation $\textcircled{I}$ can be written (with respect to a set of coordinates $x_1,\dots,x_n$) in the form :

$$\textcircled{II} \quad x_i = \sum_{j=1}^{n} \mathscr{a}_{ij}(x,t)\ \dot{b}_j$$

where $(\mathscr{a}_{ij}(x,t))$ is the matrix of $\lambda_c^{-1} \circ \lambda_b$. The analytic dependence of the right side on the initial conditions $a = x(0)$ and $t$ shows that we have solutions of $\textcircled{I}$ for $a'$ in a neighborhood $V'$ of $a$ and for all $|t| < \epsilon$ ( where $b = b(t)$ is a fixed function of $t$.) That is to say, $b(t) \circ a'$ is defined for $t$ small and $a'$ near $a$. We would like to get this result for such $a'$ and $b(t)$ running over a neighborhood of 1. To do this, consider radial paths $b(t) = t\beta$ . Then the right-side of $\textcircled{II}$ depends

analytically on the parameter $\dot{b}_j = \beta_j$; and we have solutions of the type we want for $\dot{b}$ in a neighborhood $\mathcal{V}$ of 0 in $\dot{P}_1$; i.e. b in a neighborhood V of $1$.

We now show that the solution for c in $\textcircled{I}$ (or for x in $\textcircled{II}$) with initial condition c(0) = a is a solution of $\textcircled{I}$. We apply the uniqueness theorem for ordinary differential equations

Consider the two one parameter families of transformations $b(t) \circ a$ and $c(t)$. We know that they represent the same transforamtion for t = 0 ( in solving $\textcircled{I}$ we may impose the boundary condition c(0) = a) and that their logarithmic derivatives are equal for each $t = t_0$. Let $(x^1, \ldots, x^n)$ be a coordinate system on E. Let X denote the map $p \longrightarrow (x^1(p), \ldots, x^n(p))$ of $E \subset M$ into $R^n$. We now show that for any q in E the paths $X[c(t)](q))$ and $X([b(t) \circ a]q)$ in $R^n$ satisfy the same ordinary differential equation namely :

Let $c^{\#}(t)$ denote the logarithmic derivative of the family c(t) i.e.

$$c^{\#}(t_0) = \frac{d}{dt}\Big|_{t=t_0} \widetilde{c(t_0)}^{-1} \; \widetilde{c(t)}$$

We then have

$$\frac{d}{dt} \tilde{c}(t) = \tilde{c}(t) \; c^{\#}(t)$$

Let $u(t) = X([c(t)]q)$. Then

$$\frac{du^i}{dt} = \frac{d}{dt}\, x^i([c(t)]q) = [c^{\#}(t)(x^i \circ c(t))]$$

$$= [c^{\#}(t)\, x^i]\,(c(t)q)$$

$$= A^i(t, c(t)q)$$

That is :

$$\text{(II)} \quad \frac{du^i}{dt} = B^i(t,u) \qquad \text{for } i = 1, 2, \ldots, n.$$

If $v(t) = X(b(t) \circ a(q))$ we readily verify that also $\frac{dv^i}{dt} = B^i(t,u)$ and we conclude that $b(t) \circ a(q) = c(t)q$ for all $t$ and $q$ - that is to say, $b(t) \circ a = c(t)$ for all $t$.

One sees by comparing logarithmic derivatives as above that if $X \in \mathcal{L}$ there is a path $c(t)$ in $P$ such that $\mathcal{Q}_{c(t)} = \exp tX$ on $F$. The resulting map $\rho : X \longrightarrow c(1)$ is the desired map $\rho$ of the theorem. Thus we have the analytic map $\rho : \mathcal{V} \longrightarrow P$ where $\mathcal{V}$ is a neighborhood of $0$ in the linear space $\mathcal{L}$ having the properties listed in lecture V. The diagram below is commutative and we have the existence of inverses and that $\sigma : x \longrightarrow x^{-1}$ is analytic - since the other maps in the diagram are :

$$\begin{tikzcd}
X \arrow[d, dashed] & \in & \mathcal{V} \arrow[r, "\rho"] \arrow[d, "-1"'] & P \arrow[d, "\sigma"] & \ni & \exp X \arrow[d, dashed] \\
-X & \in & \mathcal{V} \arrow[r, "\rho"] & P & \ni & (\exp X)^{-1}
\end{tikzcd}$$

## Note

The proof of the existence and analyticity of the map $\sigma$ depends only on the fact that $\mathcal{L}_c$ is independent of $c \in P$ and thus is a consequence of the pseudogroup property 1 : that $c \longrightarrow c \circ a$ is a bianalytic map of a neighborhood of the identity onto a neighborhood of a. cf. Remark following the definition of pseudogroup in lecture IV.

# LECTURE VII

## DEFINITION

Let V be a vector space over R and $X \in \mathrm{Hom}_R(V,V)$. We define $e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!}$ . Thus $e^X$ is a well-defined non-singular endomorphism (automorphism) of V ( see Chevally, I, p.5).

V can be considered as an analytic manifold and as before, we identify $\dot{V}_p$ with V by $\dot{V}_p \ni \alpha \longrightarrow \alpha' \in V$ where $\alpha(\lambda) = \lambda(\alpha')$ for every $\lambda \in \mathrm{Hom}_R(V,R)$. Then the endomorphism X defines an infinitesimal transformation $X^*$ on V : for each point $p \in V$, $X^*(p)$ is the tangent vector identified with X(p). We note that :

$$e^X = \exp X^* \text{ for } X \in \mathrm{Hom}_R(V,V).$$

To verify this, one merely notes that

$$\frac{d}{dt} e^{tX}(p) = X e^{tX}(p) = X^*(e^{tX}(p))$$

so that $e^{tX}(p)$ is the trajectory to $X^*$ with origin p.

## ROOF OF LIE'S SECOND FUNDAMENTAL THEOREM - PART I.

We assume that $\mathcal{L}$ is a linear family of infinitesimal transformations and exp $\mathcal{L}$ is a pseudogroup . We show that $[\mathcal{L},\mathcal{L}] \subset \mathcal{L}$

Note:

Keep in mind that $\exp \mathcal{L}$ means $\{\exp X;\ X \in \mathcal{L}$ and $\exp X$ has non-empty domain of definition$\}$ Since $\mathcal{L}$ is finite dimensional there always exists a neighborhood U of 0 in $\mathcal{L}$ and a non-empty open set E in M such that exp X is defined in E for all X in U.

Let X and Y be elements of $\mathcal{L}$ and consider the element of $L = \exp \mathcal{L}$ given by $\exp t\,X \exp t\,Y \exp -t\,X \exp -t\,Y$. Since L is a pseudogroup we have an analytic path $Z(t)$ in $\mathcal{L}$ such that

$$\exp t\,X \quad \exp t\,Y \quad \exp -t\,X \quad \exp -t\,Y \ = \exp Z(t)$$

on some fixed open set E in M.

As $Z(0) = 0$ we may write $Z(t) = t\,W(t)$ with $W(t)$ analytic in t. By expanding the series for the transposes $\widetilde{\exp\, t\,X}$, $\widetilde{\exp\, t\,Y}$, $\widetilde{\exp\, t\,W(t)}$ we can show that $W(0) = 0$. The computation is as follows :

$$(I-tY+\frac{t^2Y^2}{2!}-\ldots)(I-tX+\frac{t^2X^2}{2!})\ (I+tY+\frac{t^2Y^2}{2!}+\ldots)(I+tX+\frac{t^2X^2}{2!}+\ldots) =$$

$$[I+tW(t)+\frac{t^2W^2(t)}{2!}+\ldots]$$

$$I+t^2(YX-XY)+t^3(\qquad\quad) = I+tW(t)+\frac{t^2W^2(t)}{2!}+\ldots$$

$$t^2([Y,X]+t(\qquad\quad)) = tW(t)+\frac{t^2W^2(t)}{2!}+\ldots$$

$$0 = W(0)$$

We then set $W(t) = t\,\mathcal{U}(t)$ where $t \longrightarrow \mathcal{U}(t)$ is an analytic path in $\mathcal{L}$, and we have

$$t^2[X,Y] + t^3A = t^2\,\mathcal{U}(t) + t^3B.$$

Setting $t = 0$ we get $[X,Y] = \mathcal{U}(0) \in \mathcal{L}$ on the open set E and hence throughout M by analyticity.

## PROOF OF LIE'S SECOND FUNDAMENTAL THEOREM - PART II

We have the linear family of infinitesimal transformations $\mathcal{F}$ on the manifold M and assume that $[\mathcal{F}, \mathcal{F}] \subset \mathcal{F}$. Let U be a neighborhood of 0 in $\mathcal{F}$. We will prove that $\exp U = \{\exp X = X \in U\}$ is a pseudogroup of transformations in M for a suitably chosen U. Indeed, there is a non-empty open set E in M on which exp X is always defined for $X \in U$ provided U is sufficiently small. We have the mapping $\mathcal{U} : U \times E \longrightarrow M$ given by $\mathcal{U}(X,p) = \exp X(p)$ for $X \in U$ and $p \in E$. Thus we have the analytic family $(\mathcal{F}, M, \mathcal{U}_u)$, which we will show to be a pseudogroup of transformations in M after imposing one more restriction on U.

We will make use of the First Fundamental Theorem of Lie and show that logarithmic differentiation $X \longrightarrow \lambda(X)$ is one-to-one for $X \in U$ and that $\mathcal{L}_0 = \lambda(\dot{\mathcal{F}}_0)$ is the same as $\mathcal{L}_X = \lambda(\dot{\mathcal{F}}_X)$ for all X in some neighborhood of 0.

A- The mapping $\lambda : \dot{U}_0 \longrightarrow \mathcal{L}_0$ is one-to-one

Let $\alpha \in \dot{U}_0$ and let $g(t)$ be a path in U such that $g(0) = 0$ and $\dot{g}(0) = \alpha$

Then:

$$\lambda(\alpha) = \widetilde{\exp}\, g(0)^{-1} \left.\frac{d}{dt}\right|_{t=0} \widetilde{\exp}\, g(t) = \left.\frac{d}{dt}\right|_{t=0} \widetilde{\exp}\, g(t)$$

Now $\widetilde{\exp}\, g(t) = I + g(t) + \frac{g(t)^2}{2!} + \ldots$ and we can differentiate term-by-term getting :

$$\left.\frac{d}{dt}\right|_{t=0} \widetilde{\exp}\, g(t) = \dot{g}(0).$$

That is to say, $\lambda(\alpha) = \alpha$ ( remember that we identify the tangent space $\dot{V}_p$ at a point p of a vector space V with V). Thus $\lambda$ is one-to-one at 0 in U. Once we show the independence of $\mathcal{L}_X$ on $X \in U$ we will have that logarithmic differentiation is one-to-one on $\dot{U}_X$ for all $X \in U$.

B- We now show that $\mathcal{L}_X$ is independent of $X \in U$.

The family of logarithmic derivatives $\mathcal{L}_0$ consists of infinitesimal transformations of the form

$$\left.\frac{d}{dt}\right|_{t=0} (\widetilde{\exp}\, X(t)) = \dot{X}(0)$$

where $\dot{X}(0)$ denotes the tangent vector to $X(t)$ at 0.

Thus $\mathcal{L}_0 = \mathcal{F}$.

Now, for an arbitrary differentiable path $X(t)$ in U we write

$$\widetilde{\exp X} = \sum_{n=0}^{\infty} \frac{X^n}{n!}$$

Then we have

$$\frac{d}{dt}\bigg|_{t=0} (\widetilde{\exp X}) = \sum_{n=0}^{\infty} \frac{1}{n!} (X^{n-1} \dot{X} + X^{n-2} \dot{X}X + \ldots + \dot{X}X^{n-1})$$

and we set $Y = \dot{X}(t_0)$.

Consider the mappings $L_X : Y \longrightarrow XY$ and $R_X : Y \longrightarrow YX$. We may write ad $X = L_X - R_X$ or $R_X = L_X -$ ad $X$. We get the following :

$$R_X^m = (L_X - \text{ad } X)^m = \sum_{p=0}^{m} (-1)^p \binom{m}{p} L_X^{m-p} (\text{ad } X)^p$$ since $L_X$ and ad X commute

$$\begin{aligned}
\therefore \sum_{m=0}^{n} X^{n-m} YX^m &= \sum_{m=0}^{n} L_X^{n-m} R_X^m \; Y \\
&= \sum_{m=0}^{n} L_X^{n-m} \Big( \sum_{p=0}^{m} (-1)^p \binom{m}{p} L_X^{m-p} (\text{ad } X)^p \Big) (Y) \\
&= \sum_{m=0}^{n} \sum_{p=0}^{m} L_X^{n-p} (-1)^p \binom{m}{p} (\text{ad } X)^p (Y) \\
&= \sum_{p=0}^{n} \sum_{m=p}^{n} L_X^{n-p} (-1)^p \binom{m}{p} (\text{ad } X)^p (Y) \\
&= \sum_{p=0}^{n} L_X^{n-p} (-1)^p \binom{n+1}{p+1} (\text{ad } X)^p (Y) \text{ since}
\end{aligned}$$

$$\sum_{m=p}^{n} \binom{m}{p} = \binom{n+1}{p+1}$$

We then find :

$$\sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{1}{(n+1)!} X^{n-m} Y X^{m} = \sum_{n=0}^{\infty} \sum_{p=0}^{n} (-1)^{p} \frac{1}{(n+1)!} \binom{n+1}{p+1} L_X^{n-p} (\mathrm{ad}\ X)^{p} (Y)$$

$$= \sum_{n=0}^{\infty} \sum_{p=0}^{n} (-1)^{p} \frac{1}{(n-p)!(p+1)!} L_X^{n-p} (\mathrm{ad}\ X)^{p} (Y)$$

$$= \sum_{k=0}^{\infty} \frac{L_X^{k}}{k!} \sum_{p=0}^{\infty} (-1)^{p} \frac{(\mathrm{ad}\ X)^{p}}{(p+1)!} (Y)$$

$$= (\widetilde{\exp}\ X) \frac{I - e^{-\mathrm{ad}\ X}}{\mathrm{ad}\ X} (Y)$$

We then have :

$$(\widetilde{\exp}\ X)^{-1} \frac{d}{dt}\Big|_{t=0} (\widetilde{\exp}\ X(t)) = \frac{I - e^{-\mathrm{ad}\ X}}{\mathrm{ad}\ X} (\dot{X})$$

which is in $\mathcal{L}_0 = \mathcal{F}$ since we have assumed $[\mathcal{F}, \mathcal{F}] \subset \mathcal{F}$. It follows that $\mathcal{L}_X \subset \mathcal{L}_0$. Moreover $\mathcal{L}_0 \subset \mathcal{L}_X$, provided

$$\frac{I - e^{-\mathrm{ad}\ X}}{\mathrm{ad}\ X}$$

is invertible ; this is so if ad X has no eigenvalue equal $2\pi n\sqrt{-1}$ with $n \neq 0$. This proviso is satisfied for all X in some open set containing 0, and when $U$ is restricted in this way we have that $(\mathcal{F}, M, \mathcal{U}_U)$ is a pseudogroup.

Exercise:

Let D be a derivation of a Lie algebra $\mathcal{L}$.
Prove that $e^{D}$ is an automorphism of $\mathcal{L}$.

# LECTURE VIII

We will now show how to associate to an arbitrary analytic group G a Lie algebra $\dot{G}$. Using the fundamental theorems of Lie we will establish a one-to-one correspondence between the "analytic subgroups of G" and the subalgebras of $\dot{G}$ and see how certain properties of G can be studied purely in terms of $\dot{G}$. We begin by making precise what we mean by an "analytic subgroup of G".

## Definition

Given an analytic manifold M, we mean by a submanifold N of M a subset of M carrying an analytic structure such that the identity mapping $\mathcal{O}\!\ell: N \longrightarrow M$ is everywhere regular, i.e. $d\,\mathcal{O}\!\ell_p$ is a monomorphism for all $p \in N$.

## Example

Let $x^1, x^2, \ldots, x^n$ be a coordinate system on a neighborhood V of a manifold M and let $p \in V$. Let $(a^1, a^2, \ldots, a^n)$ be the coordinates of p and let $Q_\epsilon$ be a cubical neighborhood about p of semi-edge $\epsilon$ (relative to $x^1, \ldots, x^n$). Let $\xi = (\xi^{m+1}, \xi^{m+2}, \ldots, \xi^n)$ be an n-m tuple with $|\xi^i - a^i| < \epsilon$. Set $S_\xi$ equal to the subset of $Q_\epsilon$ for which $x^i = \xi^i$ $i = m+1, m+2, \ldots, n$. Then $S_\xi$ is a submanifold - it is called a "slice manifold" with respect to $x^1, x^2, \ldots, x^n$.

## Lemma 8.1

Let $N$ be a submanifold of $M$ and let $p \in N$. Then there is an open submanifold $U$ in $N$ with $p \in U$ and $U$ a slice submanifold of $M$.

## Proof

Let $y^1, \dots, y^n$ be a coordinate system at $p$ and let $\ell : N \longrightarrow M$ be the identity map. From $y^1 \circ \ell$, $y^2 \circ \ell, \dots, y^n \circ \ell$ we can select a coordinate system around $p$ in $N$ - say $y^1 \circ \ell, \dots, y^m \circ \ell$.

Let $V$ be a neighborhood of $p \in N$ so that on $V$ we have

$$y^k \circ \ell = P^k(y^1 \circ \ell, \dots, y^n \circ \ell) \text{ for each } k = m+1, \dots, n$$

Set $x^i = y^i$ for $i = 1, 2, \dots, m$ and $x^k = y^k - P^k(y^1, \dots, y^n)$ for $k = m+1, \dots, n$. Choosing a cubical neighborhood $W_\epsilon$ of $p$ on $M$ with respect to $(x^1, \dots, x^n)$ and setting $\xi = (\xi^{m+1}, \dots, \xi^n) = (0, \dots, 0)$ we get $S_\xi$ with respect to $x^1, \dots, x^n$ is a neighborhood of $p$ in $V$ for $\epsilon$ sufficiently small.

## Definition

Let $M$ be a manifold and $N$ a submanifold. Let $p$ be a point in $N$. By the tangent space to $N$ in $M_p$ we mean $d\ell_p(\dot{N}_p)$ where $\ell : N \longrightarrow M$ is the identity map.

## Proposition 8.2

Let $N$ be a submanifold of $M$ and let $X$ be an infinitesimal transformation on $M$ that is tangent to $N$, i.e. $X(p)$ is contained in $d\ell_p(\dot{N}_p)$ for each $p \in N$. If $Y$ is any other infinitesimal transformation of $M$ tangent to $N$ then $[X, Y]$ is tangent to $N$.

Proof

We define two vector fields $X^N$ and $Y^N$ by :

$$X^N(p) = d\,\mathcal{U}^{-1}_p(X(p)) \quad \text{and} \quad Y^N(p) = d\,\mathcal{U}^{-1}_p(Y(p))$$

We see that $X^N$ and $Y^N$ are infinitesimal transformations on N easily enough and, furthermore, are $\mathcal{U}$-related to X and Y, i.e

$$\tilde{\mathcal{U}} \circ X^N = X \circ \tilde{\mathcal{U}}$$

$$\tilde{\mathcal{U}} \circ Y^N = Y \circ \tilde{\mathcal{U}}$$

It follows that $\tilde{\mathcal{U}} \circ [X^N, Y^N] = [X,Y] \circ \tilde{\mathcal{U}}$; i.e. $d\,\mathcal{U}_p([X^N,Y^N]) =$

$[X,Y]\,(p)$.

We define a Lie subgroup H of a Lie group G as one would expect : H is a Lie group contained in G and such that the identity map $\mathcal{U}$: H ---> G is an everywhere regular analytic homomorphism. This definition amounts to the condition : H is an analytic submanifold closed under the formation of products and inverses and such that the map $\mu_H$: $H \times H \longrightarrow H$ given by $(x,y) \longrightarrow xy^{-1}$ is analytic on H.

This last condition will turn out to be a consequence of the condition that H be a submanifold with $H.H^{-1} \subset H$; however the proof, surprisingly enough, must wait until one can prove that $\mu_H$ is continuous. Namely, given a function f analytic around a

point p in H, we can express $f = \bar{f} \circ \mathcal{U}$ on an H-neighborhood V of p, where $\bar{f}$ is analytic on a G-neighborhood of p. Suppose $p = xy^{-1}$ and $\mathcal{M}_H$ is continuous at (x,y). Then $\mathcal{M}_H$ will map suitably small neighborhoods of (x,y) into V and thus $f \circ \mathcal{M}_H = \bar{f} \circ \mathcal{U} \circ \mathcal{M}_H$ on some neighborhood of (x,y); consequently $\bar{f} \circ \mathcal{M}_H = f \circ \mathcal{M} \circ (\mathcal{U} \times \mathcal{U})$ is analytic around (x,y) on H × H.

# LECTURE IX

In this chapter we define the Lie algebra $\dot{G}$ of a Lie group G. Chapter X will be devoted to establishing the one-to-one correspondence between analytic subgroups H of G and subalgebras $\mathfrak{H}$ of $\dot{G}$.

## Definition

Given a group G we define for each element $g \in G$ the mappings

$$L_g : x \longrightarrow gx = \text{left-translation by } g \quad \text{and}$$

$$R_g : x \longrightarrow xg = \text{right-translation by } g.$$

We denote the totality of all maps $\{L_g\}_{g \in G}$ by $L_G$.

Let the identity element of the Lie group G be denoted by "e" and the tangent space to the manifold G at e by "$\dot{G}_e$" in what follows. For each $X \in \dot{G}_e$ we denote by $X^*$ the logarithmic derivative of the group of transformations $L_G$ along the tangent vector X. We denote the family of all logarithmic derivatives by $\dot{G} = \{X^*; X \in \dot{G}_e\}$. We let $\lambda$ denote the mapping $\lambda : \dot{G}_e \longrightarrow \dot{G}$ defined by $\lambda(X) = X^*$.

Now G is a linear family of infinitesimal transformations on G and Lie's second fundamental theorem will guarantee that it is a Lie algebra once we show that exp $\dot{G}$ is a pseudogroup of transformations on G. We will now show that this is the case.

## Lemma 9.1

$X^*(e) = X$ for all $X \in \dot{G}_e$.

Proof.

To compute $X^*$ we take a differentiable path $g(t)$ in G with $g(0) = e$ and tangent vector $\dot{g}(0) = X$. Then we have for any function f analytic at e :

$$\begin{aligned} X^* &= \tilde{g}(0)^{-1} \left.\frac{d}{dt}\right|_{t=0} \tilde{g}(t) \\ X^* f(e) &= \left.\frac{d}{dt}\right|_{t=0} (\tilde{g}(t)\, f(e)) \\ &= \left.\frac{d}{dt}\right|_{t=0} f \circ g(t)\, (e) \\ &= \left.\frac{d}{dt}\right|_{t=0} f(g(t)e) \\ &= \left.\frac{d}{dt}\right|_{t=0} f(g(t)) \quad = \quad Xf \end{aligned}$$

It follows from Lemma 9.1 that the mapping $\lambda : X \longrightarrow X^* = \lambda(X)$ is an isomorphism of $\dot{G}_e$ onto G. As the side condition in Lie's first fundamental theorem is satisfied here (i.e. $X \longrightarrow \lambda(X)$ is one-to-one for $X \in \dot{G}_e$), we may apply this theorem and assert :

[illegible] : there is a neighborhood U of 0 in $\dot{G}$, a regular analytic mapping $p : U \longrightarrow G$, and an open set E containing the identity element of G such that

$$\exp X^* = L_{p(X^*)} \qquad \text{on } E$$

for all $X^* \in U$. Inasmuch as $\rho(U)$ is an open neighborhood of the identity of the analytic group G, the analytic family $(U, G\, \mathcal{C})$ defined by $\mathcal{C}(X^*, p) = \exp X^*(p)$ is indeed a pseudogroup. It follows now by Lie's second fundamental theorem that

$$[\dot{G}, \dot{G}] \subset \dot{G}$$

We call G the Lie algebra of G.

We turn out attention to the map $\rho$ above. No generality is lost is assuming that the neighborhood U contain $tX^*$, $0 \leq t \leq 1$ whenever $X^* \in U$.

Lemma 9.2

For any $X^* \in U$, $\exp X^*$ has G as its domain.

Proof

We have $\exp tX^* = (L_{\rho(tX^*)}$ on E, $0 \leq t \leq 1$. Thus $t \longrightarrow \rho(tX^*)\ (p)$ is a trajectory to $X^*$ with origin p for all $p \in E$. By analyticity, the same holds for all $p \in G$. Thus

$$\exp tX^* = L_{\rho(tX^*)}$$

throughout G.

Lemma 9.3

$\exp X^*$ is defined for all $X^* \in G$ and has domain G.

Proof

We have only to use the formula $\exp\left(\frac{X^*}{n}\right)^n = \exp X^*$. as $\frac{X^*}{n}$ will be in U for n large enough, we have $\exp \frac{X^*}{n}$ defined on all of G and hence $\left(\exp \frac{X^*}{n}\right)^n = \exp X^*$ defined on all of G.

From the discussion above we see that $\exp X^* = L_g$ for some $g \in G$. As $\exp X^*(p) = g.p$ we have, in particular, $\exp(X^*)(e) = g$.

Set $\operatorname{Exp} X^* = \exp X^*(e)$. Then $\exp X^* = L_{\exp X^*(e)} = L_{\operatorname{Exp} X^*}$ where $\operatorname{Exp} : \dot{G} \longrightarrow G$ is analytic, regular at 0, and hence bianalytic on a neighborhood of 0 in $\dot{G}$.

The mapping Exp is called the exponential mapping of the Lie algebra of G into G

# LECTURE X

We fix an analytic group G. For an ~~arbitrary~~ analytic subgroup H of G we consider $\lambda(\dot{H}_e) \subset \dot{G}$. This makes sense because $\dot{H}_e$ is canonically identified with $d\,\mathcal{C}_e(\dot{H}_e) \subset \dot{G}_e$ where $\mathcal{C}: H \longrightarrow G$ is the identity mapping. We will prove that $H \longrightarrow \lambda(\dot{H}_e)$ is a one-to-one correspondence between the analytic subgroups of G and the Lie subalgebras of $\dot{G}$.

The infinitesimal transformations of $\dot{G}$ can be characterized as the infinitesimal transformations on G which are invariant under the right translations $R_G$. The analytic subgroups H is the maximal manifold containing the identity element whose tangent space at a point g is the subspace $dR_g(\dot{H}_e) = dL_g(\dot{H}_e)$

We will divide this chapter into several parts, as follows :

§1. $\lambda(\dot{H}_e)$ is a Lie subalgebra of $\dot{G}$

§2. We define, for given subalgebra $\mathcal{H} \subset \dot{G}$, a subset H and a topology which we later prove to make H an analytic subgroup of G such that $\lambda(H) = \mathcal{H}$.

§3. We prove that H topologized as above is a topological group.

§4. The analytic structure in H.

§5. The correspondence is one-to-one.

§ 1.

Let H be an analytic subgroup of G. Then H × G is an analytic submanifold of G × G and the restriction of the map $\mu: G\times G \longrightarrow G$ defined by $\mu(g_1, g_2) = g_1 g_2^{-1}$ to the submanifold H × G is analytic. It results that $L_H$ is an analytic group of transformations on G.

Applying Lie's first fundamental theorem we find that the logarithmic derivatives to $L_H$ form a linear family independent of points in H. Hence, by Lie's second fundamental theorem, $\lambda(\dot{H}_e)$ is closed under the Poisson bracket operation and is a Lie subalgebra of $\dot{G}$.

§ 2.

Let $\mathfrak{H}$ be a Lie subalgebra of $\dot{G}$. By Lie's second theorem there is a neighborhood U of 0 in $\mathfrak{H}$ such that exp U is a pseudogroup. That is to say; there is a neighborhood U of 0 in $\mathfrak{H}$ and an open set F in G such that :

for any $a \in \exp U$, there is a neighborhood W of 0 such that $b \circ a = c$ on F where $b \in \exp W$ and $c \in \exp U$.

In this case we have $\exp X = L_{\exp X(e)} = L_{\mathrm{Exp}\, X}$ for all $X \in \dot{G}$ and we reformulate the condition that $\exp \mathfrak{H}$ is a pseudogroup as follows :

Given any neighborhood U of 0 in $\mathfrak{H}$, and $a \in \mathrm{Exp}\, U$, there is a neighborhood W of 0 in $\mathfrak{H}$ such that $\mathrm{Exp}\, W \cdot a \subset \mathrm{Exp}\, U$.

Let H be the subgroup of G generated by Exp $\mathfrak{H}$. We define as a neighborhood system of $g \in H$ the subsets Exp W.g where W ranges over neighborhoods of 0 in $\mathfrak{H}$; i.e. we take {Exp W} as a base of neighborhoods of e. To prove that this makes H a topological group we must show the following :

1) $\bigcap_W \{\text{Exp } W\} = \{e\}$

2) Given U and V, there exists W such that

   Exp W $\subset$ (Exp U) $\cap$ (Exp V).

3) Given U, there exists V such that Exp U $\supset$ (Exp V)(Exp V)$^{-1}$

4) Given U and $a \in$ Exp U, there exists V such that

   (Exp V) a $\subset$ Exp U

5) Given U and $a \in H$, there exists V such that

   $a^{-1}$(Exp V)a $\subset$ Exp U.

Note 1:

H is the same as the group generated by Exp U where U is any neighborhood of 0 in $\mathfrak{H}$ in view of the formula Exp X = $(\text{Exp}(\frac{1}{n} X))^n$

Note 2:

Notice that we do not take the relative topology for H but one which is stronger; i.e. $\iota : H \longrightarrow G$ is continuous (where $\iota$ is the identity map.)

## § 3.

Conditions 1) - 4) are easily verified, condition 4) being the statement that exp $\mathfrak{H}$ is a pseudogroup. Condition 5) is more difficult to check and requires us to study $\dot{G}$ more closely. What we will eventually show is that for $a \in G$ and $X \in \dot{G}$ we have $a^{-1}$(Exp X)a = Exp Y where Y is the image of the infinitesimal transformation X

under the inner automorphism $\Gamma_a : g \longrightarrow a^{-1}ga$ of G. Moreover $Y = \lambda(d\,\Gamma_a(\lambda^{-1}(X))$; i.e. an inner automorphism of G produces an automorphism of $\dot{G}$ more or less equivalent to the linear map $d\,\Gamma$.

### Lemma 10.1

Let $T : M \longrightarrow N$ be a bianalytic mapping. Let X be an infinitesimal transformation on M and Y an infinitesimal transformation on N. Assume, furthermore, that Y is the image of X under T. Then we have:

$$T \circ \exp sX = \exp sY \circ T$$

the domains of both sides being the same.

### Proof

In what follows we will sometimes use the notation $X_p$ for $X(p)=$ the tangent vector at p assigned by the vector field X.

The hypothesis means that $dT_p(X_p) = Y_{T(p)}$. Now for any function, f, analytic in N we have

$$Y_{T(p)}f = X_p(f \circ T).$$

In other words Yf is a function on N and $Yf \circ T = X(f \circ T)$ In terms of the transpose this last relation may be written :

$$\hat{T}Yf = X\hat{T}f$$

$$\hat{T}Y = X\hat{T}$$

i.e., X and Y are T-related.

Now

$$Y = \tilde{T}^{-1}X\tilde{T}$$

$$Y^2 = Y.Y = \tilde{T}^{-1}X\tilde{T}\tilde{T}^{-1}X\tilde{T} = \tilde{T}^{-1}X^2\tilde{T}$$

$$\vdots$$

$$Y^n = \tilde{T}^{-1}X^n\tilde{T}$$

$$\vdots$$

Summing over n we have :

$$\frac{s^nY^n}{n!} = \tilde{T}^{-1}\frac{s^nX^n}{n!}\tilde{T}$$

$$\widetilde{\exp}\, sY = \tilde{T}^{-1}\,\widetilde{\exp}\, sX\tilde{T}$$

$$\tilde{T}\,\widetilde{\exp}\, sY = \widetilde{\exp}\, sX\tilde{T}$$

$$\exp sY \circ T = T \circ \exp sX$$

An element of $\dot{G}$ is an infinitesimal transformation on G of a very special kind. The next lemma characterizes ~~these~~ vector fields on G which are elements of $\dot{G}$ as the right-invariant vector fields.

Lemma 10.2

For any $g \in G$ and $X^* \in G$ we have

$$dR_g(X^*_p) = X^*_{pg}$$

Proof

$X^*$ has the form $\lambda(X)$ where $X \in \dot{G}_e$. Let a(t) be a differentiable path in G with $a(0) = e$ and $\dot{a}(0) = X$. Then

$$\lambda(X) = \tilde{L}^{-1}_{a(0)}\,\frac{d}{dt}\Big|_{t=0}\tilde{L}_{a(t)} = \frac{d}{dt}\Big|_{t=0}\tilde{L}_{a(t)}$$

What we must show is :

$$X^* = \tilde{R}_g^{-1} X^* \tilde{R}_g$$

Now $L_a$ and $R_b$ commute for any a and b in G. So $\tilde{L}_a$ and $\tilde{R}_b$ commute. Thus $\tilde{R}_g^{-1} \tilde{L}_{a(t)} \tilde{R}_g = \tilde{L}_{a(t)}$. As this remains true in the limit we have the desired result. Taking $p = e$ we have $X^*_g = dR_g(X^*_e) = dR_g(X)$ which shows that $X^*$ depends only on its value at e. It follows that a right-invariant vector field X is in $\dot{G}$ and is $\lambda(X_e)$.

__Definition__

Given an element $g \in G$ we have the bianalytic automorphism $\Gamma_g : G \longrightarrow G$ defined by $\Gamma_g : x \longrightarrow gxg^{-1}$. Associated to this inner automorphism $\Gamma_g$ there is the mapping $\mathrm{Adg}: \dot{G} \longrightarrow \dot{G}$ defined by :

$$\mathrm{Adg} : X \longrightarrow \tilde{L}_g^{-1} X \tilde{L}_g = \tilde{\Gamma}_g^{-1} X \tilde{\Gamma}_g$$

where $\tilde{L}_g$ can be replaced by $\tilde{\Gamma}_g = \tilde{R}_g^{-1} \tilde{L}_g$ since X is right invariant.

For $X \in \dot{G}$ we define $\mathrm{ad}X: \dot{G} \longrightarrow \dot{G}$ by $\mathrm{ad}\, X: Y \longrightarrow [X,Y]$.

__Lemma 10.3__

$$\mathrm{Ad}\,\mathrm{Exp}\, X = \Sigma \frac{(-\mathrm{ad}X)^n}{n!} = e^{-\mathrm{ad}\, X}$$

for all $X \in \dot{G}$.

Proof

We consider the following ~~two~~ families of analytic transformations on $\dot{G}$ :

1) $(R, \dot{G}, \mathcal{U})$ where $\mathcal{U}(t,Y) = \mathrm{Ad\ Exp}\ tX(Y)$

2) $(R, \dot{G}, \mathcal{V})$ where $\mathcal{V}(t,Y) = e^{\mathrm{ad}\ tX}(Y)$

We easily check that $\mathcal{U}_0 = \mathcal{V}_0$, so we will know that the two families are the same if we can show that the logarithmic derivatives $\mathcal{U}^{\#}(t) = -\mathcal{V}^{\#}(t)$ for all t.

Now $\mathcal{U}^{\#}(t) = \tilde{\mathcal{U}}_t^{-1} \frac{d}{dt} \tilde{\mathcal{U}}_t$

$$= \mathrm{Ad\ Exp}\ \widetilde{tX}^{-1} \lim_{\Delta t \to 0} \frac{\mathrm{Ad\ Exp}\widetilde{(t+\Delta t}\,)X - \mathrm{Ad\ Exp}\ \widetilde{tX}}{\Delta t}$$

and since $\mathrm{Ad}\ g_1 g_2 = \mathrm{Ad} g_1 \circ \mathrm{Ad} g_2$ and $\mathrm{Exp}(t+\Delta t)X = \mathrm{Exp}\ tX \circ \mathrm{Exp}\ \Delta tX$

$$\mathcal{U}^{\#}(t) = \lim_{\Delta t \to 0} \frac{\mathrm{Ad\ Exp}\ \widetilde{\Delta tX} - I}{\Delta t}$$

Restricting to linear functions, $\lambda$, on $\dot{G}$ we have :

$$\mathcal{U}^{\#}(t) = \lim_{\Delta t \to 0} \frac{\lambda \circ \mathrm{Ad\ Exp}\ \Delta tX - \lambda}{\Delta t}$$

$$= \lambda \circ \lim_{\Delta t \to 0} \left( \frac{\mathrm{Ad\ Exp}\ \Delta tX - I}{\Delta t} \right) \quad \left( \begin{array}{l} \text{since } \lambda \\ \text{is linear} \end{array} \right)$$

$$= \lambda \left( \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}\ \ \mathrm{Exp}\ tX. \right)$$

We notice that the above shows the logarithmic derivative at t coincides with its value at 0, because we have an abelian group of transformations - not just an arbitrary family.

$$\text{Now Ad Exp } tX\,(Y) = L^{-1}_{\widetilde{\text{Exp } tX}}\; Y\, L_{\widetilde{\text{Exp } tX}}$$

$$= \exp \widetilde{tX}^{-1}\; Y \exp \widetilde{tX}$$

$$= \left(I - tx + \frac{t^2x^2}{2!} \cdots\right) Y \left(I + tx + \frac{t^2x^2}{2!} \cdots\right)$$

$$= Y + t(YX - XY) + 0(t)$$

Therefore,

$$\left(\frac{d}{dt}\bigg|_{t=0} \text{Ad Exp } tX\right)(Y) = (YX - XY) = [Y,X]$$

$$\text{and} \quad \mathcal{U}^{\#}(t)\,\lambda = -(\widetilde{adX})\,\lambda \quad \text{for any } \lambda \text{ linear on } \dot{G}$$

Identifying the tangent space at 0 in $\dot{G}$ with $\dot{G}$ itself, in the canonical way, we see that $\mathcal{U}^{\#}(t)$ is the vector field $Y \longrightarrow -\text{ad } X(Y)$.

For any linear function $\lambda$ we have

$$\psi^{\#}(t)\,\lambda = \frac{d}{dt}\bigg|_{t=0} e^{\,\widetilde{\text{ad } tX}}\,\lambda$$

$$= \widetilde{\text{ad } X}\,\lambda$$

Therefore $\mathcal{U}^{\#}(t) = -\ \psi^{\#}(t))$ for all t and $\mathcal{U}(t) = \psi(-t)$ for all t.

The lemma follows by setting t = 1.

Corollary 10.4

The notation being as above we have

$$\mathrm{Adg}(\mathfrak{H}) \subset \mathfrak{H}$$

for all $g \in H$.

Proof.

It suffices to prove this for a set of generators of $H$. Such a set may be found by taking the family $\{\mathrm{Exp}\ \mathfrak{H}\}$ of elements of $H$.

Then $\mathrm{Ad}\ \mathrm{Exp}\ X\ (\mathfrak{H}) = e^{-\mathrm{ad}\ X}(\mathfrak{H})$. As $\mathfrak{H}$ is a subalgebra we know $[\mathfrak{H}, X] \subset \mathfrak{H}$ whenever $X \in \mathfrak{H}$. It follows that $\mathrm{ad}\ X(\mathfrak{H}) \subset \mathfrak{H}$ and $\mathrm{Adg}\ (\mathfrak{H}) \subset \mathfrak{H}$ for $g \in H$.

We can define this corollary a bit more and prove as an additional corollary that condition 5) is satisfied for $H$. We first prove the following :

Lemma 10.5

Let $\alpha$ be a bianalytic automorphism of the analytic group $G$. Let $\dot{\alpha} = d\alpha\,|_e$. Then $\alpha$ maps the infinitesimal transformation $\lambda\ (X)$ into $\lambda(\dot{\alpha}\ (X))$.

Proof.

We want to prove that $\tilde{\alpha}^{-1}\ \lambda(X)\,\tilde{\alpha} = \lambda(\dot{\alpha}(X))$ for $X \in \dot{G}_e$.

First we prove $\tilde{\alpha}^{-1}\ \lambda(X)\,\tilde{\alpha}$ is again in $\dot{G}$ by showing that it is right invariant.

From $\alpha(pg) = \alpha(p)\,\alpha(g)$ it follows that $\alpha \circ R_g = R_{\alpha(g)} \circ \alpha$.

Thus $\tilde{\alpha}\tilde{R}_{\alpha(g)} = \tilde{R}_g\tilde{\alpha}$ ; $\tilde{\alpha}\tilde{R}_g = \tilde{R}_{\alpha^{-1}(g)}\,\tilde{\alpha}$; and $\tilde{\alpha}^{-1}\tilde{R}_{\alpha^{-1}(g)} = \tilde{R}_g\,\tilde{\alpha}^{-1}$.

Then:

$$\tilde{R}_g \circ \tilde{\alpha}^{-1}\;\lambda(X)\;\tilde{\alpha}\,\tilde{R}_g = \tilde{\alpha}^{-1}\,\tilde{R}^{-1}_{\alpha^{-1}(g)}\;\lambda(X)\;\tilde{R}_{\alpha^{-1}(g)}\;\tilde{\alpha}$$

$$= \tilde{\alpha}^{-1}\;\lambda(X)\;\tilde{\alpha}$$

Since $\tilde{\alpha}^{-1}\;\lambda(X)\;\tilde{\alpha}$ is in $\dot{G}$ it is of the form $\lambda(Y)$ for some Y in $\dot{G}_e$. Now $\tilde{\alpha}^{-1}\;\lambda(X)\;\tilde{\alpha}$ evaluated at e is given by :

$$\tilde{\alpha}^{-1}\lambda(X)\;\tilde{\alpha}(e) = d\alpha_e(\lambda(X)_e) = d\alpha_e(X).$$

Thus $\tilde{\alpha}^{-1}\;\lambda(X)\;\tilde{\alpha} = \lambda(\dot{\alpha}(X))$

As a corollary of Lemma 10.1 we have the formula :

$$\alpha\,(\mathrm{Exp}\ X) = \mathrm{Exp}\ \ \lambda(\dot{\alpha}(X)).$$

Corollary 10.6

Given U, a neighborhood of 0 in $\mathfrak{H}$ and given $g \in H$ there exists a neighborhood V of 0 in $\mathfrak{H}$ such that $g\ \mathrm{Exp}\ Vg^{-1} \subset \mathrm{Exp}\ U$.

Proof.

We have $\alpha(\mathrm{Exp}\ X) = \mathrm{Exp}\ \ \lambda(\dot{\alpha}(X))$. The mapping $\bar{\alpha}: X \longrightarrow \tilde{\alpha}^{-1}\ X\ \tilde{\alpha}$ is a linear mapping of $\dot{G} \longrightarrow \dot{G}$ and is therefore continuous at 0 in $\mathfrak{H}$.

Given $U'$, a neighborhood of 0 in $\dot{G}$, there exists a neighborhood $V'$ of 0 in $\dot{G}$ such that $\tilde{\alpha}^{-1}\ V'\ \tilde{\alpha} \subset U$.

If we also have $\alpha(\mathfrak{H}) \subset \mathfrak{H}$ then we find :

$$\alpha(V' \cap \mathfrak{H}) \subset U' \cap \mathfrak{H}$$

and $$\operatorname{Exp}(\alpha(V' \cap \mathfrak{H})) \subset \operatorname{Exp}(U' \cap \mathfrak{H}).$$

Taking $\alpha = \Gamma_g$ for $g \in H$ we have the desired result.

## §4.

The analytic structure in H is defined as follows - keeping in mind that $\operatorname{Exp} : \dot{G} \longrightarrow G$ is a one-to-one bianalytic map on a suitable neighborhood of 0 in $\dot{G}$ and that $\operatorname{Exp} \mathfrak{H}$ is an (analytic) pseudogroup.

Definition

Let f be a function defined around g in H. We say f is analytic at g in case $f \circ L_g \circ \operatorname{Exp}$ is analytic at 0 in $\mathfrak{H}$.

It can be shown directly that with this structure H is an analytic submanifold of G and $\lambda(\dot{H}_e) = \mathfrak{H}$. Thus H is an analytic submanifold closed under the formation of products and inverses , and such that the mapping $\mu_H : (x,y) \longrightarrow xy^{-1}$ of $H \times H$ to H is continuous. It follows from the remarks at the end of VIII that H is an analytic subgroup.

## §5.

We have shown that to every analytic subgroup H of G there corresponds a Lie subalgebra $\lambda(\dot{H}_e)$ of $\dot{G}$ and that every subalgebra

$\mathfrak{H}$ is $\lambda(\dot{H}_e)$ for some H. We now show that the correspondence is one-to-one.

Let $\lambda(\dot{H}_e) = \mathfrak{H}$. Then Exp $\mathfrak{H}$ is a neighborhood of $e \in H$. Thus $(H, G, \mathcal{A})$ is an analytic group of transformations on G where $\mathcal{A}(h,g) = h.g$. From Lie's first theorem we see that:

$$\exp : \mathfrak{H} \longrightarrow H \text{ is an anlytic map regular at } 0 \in \mathfrak{H}$$

and the correspondence is one-to-one.

# LECTURE XI

The next few lectures will be concerned with showing how certain information in G can be interpreted in $\dot{G}$ and vice-versa.

### Definition

Let $\mathfrak{g}$ be a Lie algebra. We denote $[\mathfrak{g},\mathfrak{g}]$ the smallest subspace of the underlying vector space of $\mathfrak{g}$ which contains all elements of $\mathfrak{g}$ of the form $[X,Y]$ for $X,Y \in \mathfrak{g}$. The Lie subalgebra $\mathfrak{H}$ of $\mathfrak{g}$ is called an *ideal in* $\mathfrak{g}$ in case $[\mathfrak{g},\mathfrak{H}] \subset \mathfrak{H}$; e.g. $[\mathfrak{g},\mathfrak{g}]$ is an ideal in $\mathfrak{g}$.

We call $\mathfrak{g}$ abelian when $[\mathfrak{g},\mathfrak{g}] = \{0\}$.

### Proposition 11.1

The analytic group H is abelian if and only if its Lie algebra $\dot{H}$ is abelian.

### Proof

First, assume $\dot{H}$ is abelian. For any $X,Y \in \dot{H}$ we have

$$\operatorname{Exp} X \cdot \operatorname{Exp} Y \quad (\operatorname{Exp} X)^{-1} = \operatorname{Exp} e^{0}(Y) = \operatorname{Exp} Y$$

This shows that H is generated by elements which commute, and hence H is abelian.

Assume H is abelian. For each pair of elements $X,Y \in \dot{H}$ :

$$\operatorname{Exp} e^{-\operatorname{ad} X}(Y) = \operatorname{Exp} Y$$

Therefore, $e^{-\operatorname{ad} X}(Y) = Y$ for all Y in some neighborhood of 0 in $\dot{H}$. By linearity, we conclude that this equality holds for all X,Y in in $\dot{H}$.

Since $e^{t\,\mathrm{ad}\,X}$ = the identity on $\dot{H}$ for all small t we find by differentiation :

$$\frac{d}{dt}\Big|_{t=0} e^{t\,\mathrm{ad}\,X} = \mathrm{ad}\,X = 0$$

We conclude that $\dot{H}$ is abelian.

We next show that the invariance of a subgroup H in G is equivalent to the invariance of $\dot{H}$ in $\dot{G}$ (where $\dot{H} = \lambda(\dot{H}_e) \subset \dot{G} = \lambda(\dot{G}_e)$).

Proposition 11.2

Let G be an analytic group and H an analytic subgroup of G. Let $\dot{H}$ be the subalgebra of $\dot{G}$ correspondong to H. Then H is normal in G if and only if $\dot{H}$ is an ideal in $\dot{G}$.

Proof.

H is a normal subgroup of G if and only if $\Gamma_g H = H$ for all $g \in G$. This is equivalent to Ad g $\dot{H} = \dot{H}$ for all $g \in G$. Because $\{\mathrm{Exp}\,\dot{H}\}$ is a system of generators of H, we may reformulate the condition that H is normal in G as $\mathrm{Ad}_{\mathrm{Exp}\,X}(\dot{H}) = \dot{H}$. This is the same as $e^{-\mathrm{ad}\,X}\dot{H} = \dot{H}$ for all $X \in \dot{G}$.

The last condition is satisfied if and only if we have

$$\mathrm{ad}\,X\,(\dot{H})\ \frac{d}{dt}\Big|_{t=0} e^{t\,\mathrm{ad}\,X}(\dot{H}) \subset H \quad \text{for all } X \in \dot{G}$$

and this is equivalent to $[\dot{G},\dot{H}] \subset \dot{H}$.

## Definition

An element X of the Lie algebra $\mathfrak{g}$ is called central if $[\mathfrak{g}, X] = 0$. The center of $\mathfrak{g}$ is defined to be the totality of all central elements.

## Exercise.

Show that the subgroup of G corresponding to the center of $\dot{G}$ is contained in the center of G ( it is the connected component of the center of G.)

We now state two theorems which enable us to characterize analytic groups locally by means of the Lie algebra. The proofs of these theorems will be given in lecture XII.

## Theorem 12.1

Let $\alpha : G_1 \longrightarrow G_2$ be a bianalytic isomorphism of the analytic group $G_1$ onto the analytic griup $G_2$. Then the mapping:

$$\bar{\alpha}: \dot{G}_1 \longrightarrow \dot{G}_2 \text{ defined by } \bar{\alpha}: X \longrightarrow \tilde{\alpha}^{-1} X \tilde{\alpha}$$

is an isomorphism of the Lie algebra $\dot{G}_1$ onto $\dot{G}_2$; i.c. $\tilde{\alpha}$ is linear and $\bar{\alpha}[X,Y] = [\bar{\alpha}X, \bar{\alpha}Y]$ for $X,Y \in \dot{G}_1$.

Conversely, a Lie algebra isomorphism $\bar{\rho} : \dot{G}_1 \longrightarrow \dot{G}_2$ of $\dot{G}_1$ onto $\dot{G}_2$ induces a bianalytic mapping $\rho$ of some neighborhood U of $e_{G_1}$ onto some neighborhood V of $e_{G_2}$ and $\rho(xy) = \rho(x)\ \rho(y)$ for all x and y in U. Such a map $\rho$ is called a local isomorphism of $G_1$ and $G_2$.

## Theorem 12.2

A local homomorphism $\rho : G_1 \longrightarrow G_2$ induces a homomorphism $\bar{\rho} : \dot{G}_1 \longrightarrow \dot{G}_2$ and conversely.

# LECTURE XII

Let $G_1$ and $G_2$ be analytic groups. Denote then Lie algebras by $\dot{G}_1$ and $\dot{G}_2$ respectively and let $\lambda_i (i = 1,2)$ be the logarithmic differentiation map of the tangent space, $\dot{G}_i e_{G_i}$, to $G_i$ at the identity $e_{G_i} (i = 1,2)$ respectively

Let $\rho$ be an analytic mapping of a neighborhood $U_1$ of $e_{G_1}$ into the underlying manifold of $G_2$. We define a map $\bar{\rho}: \dot{G}_1 \longrightarrow \dot{G}_2$ by

$$\bar{\rho} = \lambda_2 \circ d\rho_{e_{G_1}} \circ \lambda_1^{-1} .$$

Proposition 1 If $\rho(xg) = \rho(x)\rho(y)$ whenever $xy$, $x,y \varepsilon U_1$, then $\bar{\rho}$ is a homomorphism of the Lie algebra $\dot{G}_1$ into the Lie algebra $\dot{G}_2$.

Proof First observe that for any $X \varepsilon \dot{G}_1$, $X$ and $\bar{\rho}(X)$ are $\rho$-related; i. e., for $y$ in the domain of $\rho$.

$$d\rho_y(X(y)) = \bar{\rho}(X)(\rho(y))$$

where we denote by $X(y)$ the tangent vector at $y \varepsilon U_1$ assigned by the vector field $X$. To see this, first observe

$$X(y) = dR_y(X(e_{G_1})).$$

Then

$$\begin{aligned} d\rho_y(X(y)) &= d\rho_y \circ dR_y(X(e_{G_1})) \\ &= d(\rho \circ R_y)(X(e_{G_1})). \end{aligned}$$

Now,

$$\rho \circ R_y(x) = \rho(xy) = \rho(x)\rho(y) = R_{\rho(y)} \circ \rho(x)$$

Thus,

$$\begin{aligned} d\rho_y(X(y)) &= d(R_{\rho(y)} \circ \rho)(X(e_{G_1})) \\ &= dR_{\rho(y)} \circ d\rho(X(e_{G_1})) \\ &= dR_{\rho(y)}[\bar{\rho}(X)(e_{G_1})] \\ &= \bar{\rho}(X)(\rho(y)) \end{aligned}$$

Now, since $X$ and $\bar{\rho}(X)$ are $\rho$-related we have for each $X$ and $Y$ in $G_1$ that $[X,Y]$ is $\rho$-related to, $[\bar{\rho}(X), \bar{\rho}(Y)]$ at least on neighborhoods of $e_{G_1}$ and $e_{G_2}$. In particular,

$$(d\rho_{e_{G_1}}\{[X,Y](e_{G_1})\} = [\bar{\rho}(X),\bar{\rho}(Y)](e_{G_2}).$$

However, the left-hand side of the last equation is $\{\bar{\rho}([X,Y])\}(e_{G_2})$ and two right-invariant infinitesimal transformations which agree at the identity are equal - thus $[\bar{\rho}(X),\bar{\rho}(Y)] = \bar{\rho}[X,Y]$.

Remark. If, under the hypotheses of Proposition 1, we have that $\rho(U_1)$ is a neighborhood of $G_2$ then $\bar{\rho}$ is subjective.

Proof. $r = \text{rank } \bar{\rho} = \text{rank } d\rho_{e_{G_1}} = \text{rank } d\rho_y$ for any $y \in U_1$. Thus $\rho(U_1)$ is a neighborhood of $e_{G_2}$ if and only if $r = \dim G_2$

If the image $\rho(U_1)$ is a neighborhood $U_2$ of $e_{G_2}$ and $\rho(xy) = \rho(x)\rho(y)$ for $x,y \in U_1$, we call $\rho$ a local epimorphism.

We see that a local epimorphism $\rho : G_1 \longrightarrow G_2$ induces an epimorphism $\bar{\rho} : \dot{G}_1 \longrightarrow \dot{G}_2$. If the local epimorphism $\rho$ is injective, it is a diffeomorphism of a neighborhood $\mathcal{U}_1$ of $e_{G_1}$ onto a neighborhood $\mathcal{U}_2$ of $e_{G_2}$. Thus $\dim G_1 = \dim G_2 = \operatorname{rank} \bar{\rho}$ and $\bar{\rho}$ is thus an isomorphism of $\dot{G}_1$ onto $\dot{G}_2$.

We now discuss the converse situation. Let there be given a homomorphism $\bar{\rho} : \dot{G}_1 \longrightarrow \dot{G}_2$ of the Lie algebra $\dot{G}_1$ onto the Lie algebra $\dot{G}_2$. Let $\mathrm{Exp}_i$ $(i = 1,2)$ denote the Exponential mapping $\mathrm{Exp}_i : \dot{G}_i \longrightarrow G_i (i = 1,2)$. Let $\mathcal{V}$ be a neighborhood of 0 in $\dot{G}_1$ such that $\mathrm{Exp}_1$ is a bianalytic map of $\mathcal{V}$ onto $\mathcal{U}_1 = \mathrm{Exp}\, \mathcal{V}$ and $\mathrm{Exp}_2$ is a bianalytic map of $\bar{\rho}(\mathcal{V})$ onto $\mathcal{U}_2 = \mathrm{Exp}(\bar{\rho}(\mathcal{V}))$. Define $\rho(\mathrm{Exp}_2 X) = \mathrm{Exp}_2(\bar{\rho}(X))$ for $X \varepsilon \mathcal{V}$. Then $\rho$ is a well-defined map of $\mathcal{U}_1$ onto $\mathcal{U}_2$ and if $\bar{\rho}$ is injective so is $\rho$. Thus we will have proved the converse to Proposition 1 when we have shown that $\rho(xy) = \rho(x)\rho(y)$ for $x,y \varepsilon \mathcal{U}_1$. This will follow if we can show that $X$ and $\bar{\rho}(X)$ are $\rho$-related, for then we have:

$$(\rho \circ \exp X)(y) = (\exp \bar{\rho}(X) \circ \rho)(y)$$

and thus,

$$\rho(\mathrm{Exp}_1 X \cdot y) = (\mathrm{Exp}_2 \bar{\rho}(X)) \cdot \rho(y) \ .$$

whence

$$\rho(\mathrm{Exp}_1 X \cdot y) = \rho(\mathrm{Exp}_1 X)\rho(y).$$

Now, to prove that $X$ and $\bar{\rho}(X)$ are $\rho$-related we proceed

as follows. Writing $G$ for $G_1$ we identify $\dot{G}_{\mathrm{Exp}\,Y}$ with $\dot{G}$ by the sequence

$$\dot{G}_{\mathrm{Exp}\,Y} \xrightarrow{d\mathrm{Exp}^{-1}} \left(\overset{:}{G}\right)_Y \longrightarrow \dot{G} .$$

let $\psi(Y) = \left(\frac{I-e^{-\mathrm{ad}\,Y}}{\mathrm{ad}\,Y}\right)^{-1}$ for $Y$ in some neighborhood of $0$ in $\dot{G}$. Then, with the above identifications the vector-field $X$ is described by (cf. Lecture VII, p. 43)

$$X(\mathrm{Exp}\,Y) = [\psi(Y)](X).$$

In particular,

$$\bar{\rho}(X)(\mathrm{Exp}\,\bar{\rho}(Y)) = \psi(\bar{\rho}(Y)\cdot\bar{\rho}(X)).$$

We want to show that $d\rho_{\mathrm{Exp}\,Y}(X) = \bar{\rho}(X)(\rho(\mathrm{Exp}\,Y))$. The right-hand side of this last equation is just $\bar{\rho}(X)(\mathrm{Exp}\,\bar{\rho}(Y))$. Now $\rho = \mathrm{Exp}_2 \circ \bar{\rho} \circ \mathrm{Exp}_1^{-1}$ and $X(\mathrm{Exp}\,Y) = [\psi(Y)]X$. Then

$$\begin{aligned} d\rho_{\mathrm{Exp}\,Y}(X(\mathrm{Exp}\,Y)) &= d(\mathrm{Exp} \circ \bar{\rho} \circ \mathrm{Exp}^{-1})_{\mathrm{Exp}\,Y}(X(\mathrm{Exp}\,Y)) \\ &= d(\mathrm{Exp} \circ \bar{\rho})\,[d\,\mathrm{Exp}^{-1}{}_{\mathrm{Exp}\,Y}(X(\mathrm{Exp}\,Y))] \\ &= d(\mathrm{Exp} \circ \bar{\rho})\{[\psi(Y)]X\}. \tag{*} \end{aligned}$$

We may write

$$\bar{\rho}\,[X,Y] = [\bar{\rho}(X), \bar{\rho}(Y)];\ \text{i. e.}$$

$$\bar{\rho} \circ \mathrm{ad}\,X = \mathrm{ad}(\bar{\rho}(X)) \circ \bar{\rho}$$

$$\bar{\rho} \circ \mathrm{ad}\,Y = \mathrm{ad}(\bar{\rho}(Y)) \circ \bar{\rho}$$

and $\bar{\rho} \circ \psi(Y) = \psi(\bar{\rho}(Y)) \circ \bar{\rho}$. Therefore,

$$\bar{\rho}\{[\psi(Y)](X)\} = [\psi(\bar{\rho}(Y))](\bar{\rho}(X)). \tag{**}$$

Using the formula of (**) we see that (*) is equal to

$$\mathrm{dExp}\{[\psi(\bar{\rho}(Y))](\bar{\rho}(X))\} = \bar{\rho}(X)(\mathrm{Exp}\ \bar{\rho}(Y))$$

and we have shown that $X$ and $\bar{\rho}(X)$ are $\rho$-related.

# LECTURE XIII

We devote this lecture to a discussion of a certain important subgroup of G - the commutater subgroup [G,G]. This discussion will provide an illustration of most of what has been said in the previous lectures.

**Definition**

Let G be a Lie group and $\mathfrak{g}$ a Lie algebra. By the commutator subgroup [G,G] of G we mean the subgroup generated by all elements of the form $g_1^{-1}g_2^{-1}g_1g_2$ where $g_1, g_2 \in G$.

We call $[\mathfrak{g},\mathfrak{g}]$ the commutator subalgebra of $\mathfrak{g}$.

**Proposition 13.1**

The commutator subgroup [G,G] is normal in G. The (abstract) group G/[G,G] is abelian and if H is a normal subgroup of G such that G/H is abelian H must contain [G,G].

**Proof.**

To see that [G,G] is normal we observe :

$$g^{-1}g_1^{-1}g_2^{-1}g_1g_2g = g^{-1}g_1^{-1}g\; g^{-1}g_2^{-1}g\; g^{-1}g_1g\; g^{-1}g_2g$$

$$= (g^{-1}g_1g)^{-1}(g^{-1}g_2g)^{-1}(g^{-1}g_1g)(g^{-1}g_2g)$$

The second part of the proposition follows from the fact that G/H is abelian if and only if $g_1g_2H = g_2g_1H$ for all $g_1, g_2 \in G$ and this is equivalent to

$$g_1g_2g_1^{-1}g_2^{-1} \in H$$

**Proposition 13.2**

If $G$ is an analytic group (i.e. a connected Lie group) then $[G,G]$ is connected.

**Proof**

For locally euclidean spaces connected-ness and arc-wise connectedness are the same.

Now, for any $g_1, g_2 \in G$ let $g_1(t)$ be a path connecting $e$ to $g_1$ and $g_2(t)$ a path connecting $e$ to $g_2$. Then $g_1^{-1}(t)g_2^{-1}(t)g_1(t)g_2(t)$ is a path consisting of commutators connecting $e$ to $g_1^{-1}g_2^{-1}g_1g_2$. Thus $[G,G]$ is arc-wise connected.

**Theorem 13.3**

The commutator subgroup of an analytic group $G$ is an analytic subgroup whose Lie algebra is $[\dot{G},\dot{G}]$.

**Proof**

Let $H$ be the analytic subgroup whose Lie subalgebra is $[\dot{G},\dot{G}]$. Let $U_1 = [X_1,Y_1]$, $U_2 = [X_2,Y_2], \ldots, U_r = [X_r,Y_r]$ be a base of $[\dot{G},\dot{G}]$. Consider the map $k : R^{2r} \longrightarrow H$ defined as follows :

$$\text{Let } R^{2r} \ni (s,t) = (s_1,t_1,s_2,t_2,\ldots,s_r,t_r).$$

$$\text{Set } k_i(s,t) = \operatorname{Exp} s_iX_i \operatorname{Exp} t_iY_i \operatorname{Exp} -s_iX_i \operatorname{Exp} -t_iY_i$$

$$\text{Set } k(s,t) = \prod_{i=1}^{r} k_i(s,t).$$

We observe that $k_i(0,t) = e = k_i(s,0)$. We now show :

**Lemma**

The differential $dk_{(0,t,0,t,\ldots,0,t)}$ maps the tangent space

$$\dot{R}^{2r}_{(0,t,\ldots,0,t)} \text{ onto the tangent space } \dot{H}_e.$$

**Proof**

Let f be an arbitrary function analytic on G at e. Then :

$$\left(dk_{i(0,t)} \frac{d}{ds}\right)f = \left.\frac{d}{ds}\right|_{s=0} f(k_i(s,t))$$

$$= \left.\frac{d}{ds}\right|_{s=0} [f(\exp sX_i \exp tY_i \exp -sX_i \exp -tY_i)(e)]$$

We rewrite this :

① $$\left[\left.\frac{d}{ds}\right|_{s=0} \widetilde{\exp}\, -tY_i\ \widetilde{\exp}\, -sX_i\ \widetilde{\exp}\, tY_i \exp sX_i\ f\right](e)$$

We have seen previously that

$$\exp sX \exp Y \exp -sX = \exp(e^{-\mathrm{ad}\, sX}(Y))$$

$$= \exp\,(Y - [sX,Y] + s^2 \ldots\ldots).$$

So ① becomes :

$$\left[\left.\frac{d}{ds}\right|_{s=0} \widetilde{\exp}\, tY_i\ \widetilde{\exp}\,(tY_i + s[X_i,tY_i])f\right](e)$$

which is $\left(\dfrac{1 - e^{-\mathrm{ad}\, tY_i}}{\mathrm{ad}\, tY_i}\right)[X_i,tY_i]\, f(e).$

Set $\psi(Z) = \frac{1 - e^{-\mathrm{ad}\, Z}}{\mathrm{ad}\, Z}$ . Then $\psi$ is an analytic function of Z and $\psi(0)$ = identity. $\psi$ maps ideals into ideals ( it is a series in ad Z) and in particular $\psi : \dot{H} \longrightarrow \dot{H}$.

Thus, for t small we have that

$$dk_{(0,t)} \frac{1}{t} \frac{d}{ds_i} \text{ is in } \dot{H}_e \text{ near } U_i(e).$$

As things near a base are again a base we have :

$$I_m \; dk_{(0,t)} = \dot{H}_e$$

The lemma shows that the image of k covers a neighborhood of e in H and thus [G,G] contains H.

We now prove that H contains the commutator subgroup by proving that H contains all the commutators of a set of generators for G (any commutator is a product of commutators of a set of generators and their conjugates and since H is normal we will have $H \supset [G,G]$ ).

We know that $\dot{G}/\dot{H} = \dot{G}/[\dot{G},\dot{G}]$ is an abelian Lie algebra. Set n equal to dim $\dot{G}/\dot{H}$. Then $R^n$ is an analytic group with respect to addition and its Lie algebra $\dot{R}^n$ is isomorphic to $\dot{G}/\dot{H}$.

Let $\theta$ be a linear map of $\dot{G}$ onto the Lie algebra of $R^n$ with kernel $\dot{H}$. $\theta$ is automatically a Lie algebra homomorphism. On a neighborhood U of e in G define $\rho : U \longrightarrow R^n$ by the formula :

$$\rho\,(\mathrm{Exp}\; X) = \mathrm{Exp}\;\; \theta(X).$$

We have seen that $\rho(xy) = \rho(x) + \rho(y)$ if U is sufficiently small. Choose a symmetric neighborhood V of E in G with $VV \subset U$. Then for any x, y in V we have

$$\rho(xyx^{-1}y^{-1}) = 0 = \text{identity in } R^n$$

We can assume that $\text{Exp}^{-1}$ exists on U and then $\rho(x) = 0$ implies x is in H - since $x = \text{Exp } X$ and $\rho(\text{Exp } X) = \text{Exp } \theta(X)$ implies $\theta(X) = 0$ and X is in $\dot{H}$.

We have shown that commutators of elements of V lie in H and so all commutators lie in the normal subgroup H.

# LECTURE XIV

We now compare the two topologies on the subgroup H of G corresponding to the subalgebra $\mathfrak{H}$ of $\dot{G}$ - viz, the topology as analytic subgroup in which the family of sets {Exp U} is taken as a system of neighborhoods of e, and the relative topology on H as a subspace of G. In particular we will show that if $\mathfrak{H} \subsetneq \dot{G}$ then $H \subsetneq G$.

## Theorem 14.1

Let G be an analytic subgroup and H an analytic subgroup. There exists a coordinate system at the identity e and a cubical neighborhood $V_a$ of semi-edge a such that $H \cap V_a$ consists of a denumerable number of slice submanifolds $x^{k+1} = \text{const.}, \ldots, x^n = \text{const.}$ where k = dimension H and n = dimension G.

## Proof

Consider the map $\emptyset : R^n \longrightarrow G$ given by

$$\emptyset(s^1, s^2, \ldots, s^n) = \text{Exp } s^1X_1 \text{ Exp } s^2X_2 \ldots \text{Exp } s^nX_n$$

where $X_1, \ldots, X_n$ is a base for $\dot{G}$ such that $X_1, \ldots, X_k$ span $\dot{H}$. Since $d\emptyset\left(\frac{\partial}{\partial s^i}\right) = X_i(e)$ we have that $\emptyset$ is regular at $(0,0,\ldots,0)$. We shall show that $x^i = s^i \circ \emptyset^{-1}$ is a coordinate system at e with the properties desired.

Choose a cubical neighborhood $V_a$ with respect to $(x^1, x^2, \ldots, x^n)$. It is immediate from the definition of the mapping $\emptyset$ that each slice $x^{k+1} = \text{const.}, \ldots, x^n = \text{const.}$ lies in an H-coset. It follows that $H \cap V_a$ consists of slices; i.e. if a point of a slice is in H then the whole slice is in H.

The slices in $H \cap V_a$ are disjoint open sets in $H$ (open by the definition of $\emptyset$ and the topology in $H$ as an analytic subgroup). Since $H$ is connected we have $H = \sum_r U^r$ where $U$ is any neighborhood of $e$ in $H$. Thus $H$ satisfies the second axiom of countability and $H \cap V_a$ is the union of countably many slices.

Corollary 14.2

The open sets in $H$ (as analytic subgroup) are the connected components of open subsets in the relative topology,

Corollary 14.3

$H$ is closed if and only if $H \cap V_a$ is a single slice for a suitably small choice of $a$.

Proof

Assume $H$ is closed. Choose a closed cubical neighborhood $F$ of $e$. Theh $H \cap F$ consists of slices $S_\xi$ where $\xi \in Q^{n-k}$ (closed cube in $R^{n-k}$) and $S_\xi$ is the slice defined by $x^{k+1} = \xi^{k+1}, \dots, x^n = \xi^n$.

Now $H \cap F$ is closed just in case the set $P$ of all points $\xi$ above forms a closed subset in $Q$.

Lemma

If $\xi \in P$ and $\xi$ is an accumulation point of $P$ then each point of $P$ is an accumulation point of $P$.

Proof.

Consider the map $\varphi : H \longrightarrow H$ given by

$$\varphi : x \longrightarrow x.(\mathrm{Exp}\,\xi^{k+1}X_{k+1}\dots\mathrm{Exp}\,\xi^{n}X)^{-1}$$

The map $\mathcal{U}$ takes a neighborhood of $y = (\operatorname{Exp} \xi^{k+1}X_{k+1} \ldots \operatorname{Exp} \xi^{n}X_{n})$ to a neighborhood of e.

Let $W \subset V_a$ be a neighborhood of y with $W_y^{-1} \subset V_a$. Then

$$R_y^{-1}(H \cap W) = (R_y^{-1}W) \cap H \subset V_a \cap H$$

So we see that if the slices accumulate at y they also accumulate at e, and hence, by translation, at every point $Z \in V_a \cap H$.

From the lemma and the preceding remarks we see that if P is infinite it is both countable and perfect. As this is impossible, we conclude that P is finite. So for a smaller choice of the number a we can arrange that $V_a \cap H$ consists of a single slice. If $V_a \cap H$ is a single slice, then $\overline{H} \cap V_a = H \cap V_a$. Thus $\overline{H} = H$ ( by translating our information to a neighborhood of any point of H.)

Corollary 14.4

H is closed if and only if the relative topology coincides with its analytic group topology.

## LECTURE XV

We now take up the problem of interpreting $G/_H$ as an analytic manifold where H is a closed analytic subgroup of G.

### § 1.

Let G be an analytic group and H an analytic subgroup. Take a base $X_1, X_2, \ldots, X_n$ for $\mathring{G}$ in which $X_1, X_2, \ldots, X_k$ span $\dot{H}$. We introduce a coordinate system $(x^1, X^2, \ldots, x^n)$ at e in G such that the element g near e is given by :

$$g = \operatorname{Exp} x^1(g)X_1 \ \operatorname{Exp} x^2(g)X_2 \ \ldots \ \operatorname{Exp} x^n(g)X_n$$

If $V_\epsilon$ is a cubical neighborhood of e in G, we know from lecture XIV that the slice $x^{k+1} = \text{const.}, \ldots, x^n = \text{const.}$ is the connected component of an H-coset $H_g \cap V_\epsilon$ and conversely. We will use the following consequence in what follows :

**Corollary 15.1**

Let H be a closed analytic subgroup of G. Then there exists a positive number $\epsilon$ such that two elements in $V_\epsilon$ belong to the same H-coset if and only if they belong to the same slice $x^{k+1} = \text{const.}, \ldots, x^n = \text{const.}$

**Proof**

Choose $\epsilon_1$, such that $H \cap V_{\epsilon_1}$ consists of a single slice. Choose $\epsilon_2$ such that $V_{\epsilon_2} V_{\epsilon_2} \subset V_{\epsilon_1}$. Choose $\epsilon$ such that $V_\epsilon V_\epsilon^{-1} \subset V_{\epsilon_2}$.

Then given x and y in $V_\epsilon$ and both in the same H-coset, we may write

$$x \in Hy \cap V_\epsilon \quad .$$

Then $xy^{-1} \in H \cap V_{\epsilon_2}$ - a single slice in $V_{\epsilon_2}$. Writing $x = xy^{-1}y$ we see that $x \in (H \cap V_{\epsilon_2}).y \subset V_{\epsilon_1} \cap Hy$.

Now , $H \cap V_{\epsilon_2}$ is connected and so x is contained in the connected component of y in $Hy \cap V_{\epsilon_1}$. It follows that x and y lie in the same slice of $V_{\epsilon_1}$. Since the intersection of a slice with a smaller cubical neighborhood is again a slice we have that x and y lie in the same slice of $V_\epsilon$ .

Definition

Let G be a topological group and H a subgroup. We denote the family of left cosets by $G/_H$ and let $\pi : G \longrightarrow G/_H$ denote the canonical projection. We topologize the set $G/_H$ by taking as open sets those sets of the form $\pi(U)$ with U open in G. The mapping $\pi$ is thus continuous and open. $G/_H$ is a Hausdorff space if and only if H is closed in G. We denote by $H \backslash G$ the family of right cosets topologized in like fashion.

Definition

Let U be a neighborhood in $G/_H$. By a cross-section on U we mean a continuous map $\ell : U \longrightarrow G$ such that $\pi \circ \ell$ is the identity on U. By a local cross section at a point $p \in G/_H$ we mean a cross-section on some neighborhood U of p.

We have a natural map $\mu : G \times G/_H \longrightarrow G/_H$ defined by

$$\mu : (g, xH) \longrightarrow gxH.$$

This mapping $\mu$ is continuous and induces a mapping $\mu^H : G \longrightarrow \{$ group of homeomorphisms of $G/_H$ given by $\mu^H : g \longrightarrow L_g^H$ where $L_g^H : xH \longrightarrow gxH$. We have the commutative diagram:

$$\begin{tikzcd}
G \arrow[r, "L_g"] \arrow[d, "\pi"'] & G \arrow[d, "\pi"] \\
G/_H \arrow[r, "L_g^H"'] & G/_H
\end{tikzcd}$$

and it follows that a local corss-section at p gives rise to a local cross-section at any other point q.

Proposition 15.2

Let G be an analytic group and H a closed analytic subgroup. There is a local cross-section at the identity in $G/_H$.

Proof

Let $\epsilon$ be chosen as in Corollary 15.1. On the neighborhood $\pi(V_\epsilon)$ consider the map $\varphi : \pi(g) \longrightarrow$ point whose coordinates are $(0, 0, \ldots, 0, x^{k+1}(g), \ldots, x^n(g) \rangle$ Since points of $V_\epsilon$ in the same H-coset are in the same slice, this map is well-defined and it is continuous.

It results now that G is a fibre-bundle over $G/_H$ in the sense of Steenrod (Steenrod, Topology of Fibre Bundles, Princeton, 1951) The existence of local cooss-sections is essentially the statement $\pi^{-1}(U) \approx U{\times}H$. Globally, the situation is described by piecing together according to a structure group; viz., $L_H$ the group of left translations by elements of H. The quadruple $(G, G/_H, \pi, L_H)$ is the resulting fibre bundle.

Theorem 15.3

There exists precisely one analytic structure on $G/_H$ such that the canonical map $\mu : G \times G/_H \longrightarrow G/_H$ is analytic.

With this structure, $G/_H$ has analytic local cross-sections.

Proof

In a neighborhood U of the coset H we introduce the coordinate system $x^{k:1} \circ \mathcal{A}, x^{k+2} \circ \mathcal{A}, \ldots, x^n \circ \mathcal{A}$ where $\mathcal{A}$ is the local cross-section defined before. By translation, we get coordinate systems at each point; viz, on $L^H_g$ (U) take as coordinate system

$$x^{k+1} \circ \mathcal{A} \circ L^H_{g^{-1}}, \ldots, x^n \circ \mathcal{A} \circ L^H_{g^{-1}}.$$

We can easily verify that this is an analytic structure and the local cross-section $\mathcal{A}$ is analytic by construction. The uniqueness of the analytic structure is straight-forward.

We state now several important facts and refer the reader to the new book of Helgason (Differential Geometry and Symmetric Spaces, Interscience ) for the proof of the first

Theorem

A connected closed subgroup $H$ of an analytic group $G$ is an analytic subgroup.

Theorem

An arcwise-connected subgroup of an analytic group is analytic.

In particular, it results that the connected component of the identity in any closed subgroup is an analytic subgroup. Now, the connected component of the identity is also open and the proof of theorem 15.3 can be modified for spaces $G/_H$ where $G$ is an analytic group, $H$ a closed subgroup containing the analytic subgroup $H_1$ as an open subgroup. It follows that if $H$ is a closed subgroup of the analytic group $G$ the space $G/_H$ has an analytic structure and analytic local cross-sections.

2.

In the case where $H$ is closed and normal in $G$ we have an analytic structure on $G/_H$ and a group structure. In the following diagram all maps are already known to be analytic except $\tau$ (the multiplication in $G/_H$) :

$$\begin{tikzcd}
G/_H \times G/_H \arrow[r, "\tau"] \arrow[d, "\alpha \times \text{identity}_{G/_H}"] & G/_H \\
G \times G/_H \arrow[r, "\mu"] & G/_H \arrow[u, "\text{identity } G/_H"']
\end{tikzcd}$$

As $\tau' = \text{identity}_{G/H} \circ \mu \circ \varphi \times \text{identity}_{G/H}$ is the composition of analytic maps it is also analytic.

Thus $G/_H$ is an analytic group and $\pi : G \longrightarrow G/_H$ is an analytic homomorphism. The triple $(G, \pi, G/_H)$ enjoys the universal property :

Let $G$ and $G'$ be analytic groups and $\rho : G \longrightarrow G'$ an analytic homomorphism with $H \subset \ker \rho$ . Then there exists precisely one analytic homomorphism $\bar{\rho}$ such that the following diagram commutes:

$$\begin{tikzcd}
G \arrow[r, "\pi"] \arrow[dr, "\rho"'] & G/_H \arrow[d, dashed, "\bar{\rho}"] \\
 & G
\end{tikzcd}$$

# LECTURE XVI

In this chapter we discuss analytic homomorphisms of groups $\rho: G \longrightarrow H$ and relate them to homomorphisms $\dot\rho: \dot G \longrightarrow \dot H$.

Definition: Let $\rho: G \longrightarrow H$ be an analytic homomorphism.

We define a linear map $\dot\rho: \dot G \longrightarrow \dot H$ as follows:

$$\dot\rho(X)_h = dR_h \circ d\rho(X_{e_G})$$

where $e_G$ is the identity in $G$.

Lemma 16.1: $X$ and $\dot\rho(X)$ are $\rho$-related.

Proof: $d\rho_g(X_g) = d\rho_g \circ dR_g(X_{e_G})$.

Since $\rho(xg) = \rho(x)\rho(g)$ we have $\rho \circ R_g = R_{\rho(g)} \circ \rho$,

whence $d\rho_g \circ dR_g(X_{e_G}) = dR_{\rho(g)} \circ d\rho_{e_G}(X_{e_G})$

$$= \rho(X)_{\rho(g)}$$

Now, whenever $X_1$ is $\rho$-related to $Y_1$ and $X_2$ is $\rho$-related to $Y_2$ we have $[X_1,X_2]$ is $\rho$-related to $[Y_1,Y_2]$ for any analytic map $\rho$. In particular, we have here $\dot\rho[X,Y] = [\dot\rho(X),\dot\rho(Y)]$ and is a homomorphism of the Lie algebra $\dot G$ into the Lie algebra $\dot H$.

Furthermore, the map $\cdot$ : $\mathrm{Hom}(G,H) \longrightarrow \mathrm{Hom}(\dot G,\dot H)$ is functional; i.e., $(\rho \dot\circ \sigma) = \dot\rho \circ \dot\sigma$, as we see from the fact that $d(\rho \circ \sigma) = d\rho \circ d\sigma$.

Lemma 16.2: The mapping $\cdot$ is injective; i.e., given $\rho,\sigma \in \mathrm{Hom}(G,H)$ and $\dot\rho = \dot\sigma$ we conclude that $\rho = \sigma$.

Proof: $\rho \circ \exp X\,(e_G) = \exp \dot\rho(X) \circ \rho(e_G)$. Thus we have:

(*) $\rho(\mathrm{Exp}\ X) = \mathrm{Exp}\ \dot\rho(X)$ for each $X \in \dot G$. If $\dot\rho = \dot\sigma$ then we also have $\rho(\mathrm{Exp}\ X) = \sigma(\mathrm{Exp}\ X)$ and, since the elements $\{\mathrm{Exp}\ X\}$ generate $G$, we must have $\rho = \sigma$.

Another immediate consequence of the formula (*) is:

Note 1: $\dot{\rho}$ is one-to-one if and only if $\rho$ is one-to-one in a neighborhood of $e_G$ .

We now state and prove some other important properties of "$\cdot$" .

Note 2: The connected component of the identity of $\operatorname{Ker} \rho$ is the analytic subgroup $N$ corresponding to the subalgebra $\operatorname{Ker} \dot{\rho}$ of $\dot{G}$ .

Proof: Clearly $N$ is contained in $\operatorname{Ker} \rho$ . Now, let $U$ be a neighborhood of $e_G$ of the form $\operatorname{Exp} \mathcal{U}$ where

1) $\mathcal{U}$ is a neighborhood of $0$ in $\dot{G}$

2) $\operatorname{Exp}: \dot{\rho}(\mathcal{U}) \longrightarrow H$ is one-to-one

For $X \in \mathcal{U}$ we have

$$e_H = \operatorname{Exp} \dot{\rho}(X) \Longleftrightarrow \dot{\rho}(X) = 0 \Longleftrightarrow X \in \operatorname{Ker} \dot{\rho} \Longleftrightarrow \operatorname{Exp} X \in N$$

which tells us that $\operatorname{Ker} \rho \cap U = N \cap U$ . This yields that $N$ is open in $\operatorname{Ker} \rho$ . Since an open subgroup is also closed, it follows that $N$ is the connected component of the identity of $\operatorname{Ker} \rho$ .

Corollary: The analytic subgroup $N$ of $G$ , corresponding to the Lie subalgebra $\operatorname{Ker} \dot{\rho} \subset \dot{G}$ is closed in $G$ .

Note 3: Let $G$ be a simply connected analytic group. Then the mapping "$\cdot$" of $\operatorname{Hom}(G,H)$ into $\operatorname{Hom}(\dot{G},\dot{H})$ is bijective.

Proof:

A) We will first give a simple proof refering the reader to the book of Pontyagin for a proof of the key fact; viz. that a local homomorphism of $G$ (connected and simply connected) can be extended to a homomorphism of all of $G$ . With this tool at hand we produce the local homomorphism

$\rho(\operatorname{Exp} X) = \operatorname{Exp} \theta(X)$ for $\theta \in \operatorname{Hom}(\dot{G},\dot{H})$ . We now give a fuller treatment.

B) The analytic group $G \times H$ has a Lie algebra which may be identified with $\dot{G} \times \dot{H}$ . Set

$$\mathcal{G} = \{(X, \theta(X)); X \in \dot{G}\} .$$

$\mathcal{G}$ is a Lie subalgebra of $\dot{G} \times \dot{H}$ whose corresponding analytic subgroups will be denoted $\tilde{G}$ .

Denote by $\pi'$ and $\pi''$ the projections $\pi':G \times H \longrightarrow G$ and $\pi'':G \times H \longrightarrow H$ . Let $\sigma = \pi'|\tilde{G}$ (the restriction of $\pi'$ to the subgroup $\tilde{G} \subset G \times H$) and $\tau = \pi''|\tilde{G}$ . As $\dot{\sigma} = \dot{\pi}'|\dot{\tilde{G}}$ is one-to-one, we see that $\sigma$ is locally one-to-one on an open connected neighborhood $U$ in $\tilde{G}$ . We can now show that the pair $(\tilde{G}, \sigma)$ is a covering space of $G$ .

Lemma: Each point in $G$ has a neighborhood $V$ such that each component of $\sigma^{-1}(V)$ is mapped homomorphically onto $V$ by $\sigma$ .

Proof: Let $D = \sigma^{-1}(1) = \operatorname{Ker} \sigma$ . Let $V = \sigma(U)$ . Then $\sigma^{-1}(V) = D \cdot U = \{du;\ d \in D,\ u \in U\}$ . For $d, d' \in D$ we have $dU \cap d'U \neq \emptyset \iff d = d'$ . To see this we write:

$$du = d'u'$$
$$u = d^{-1}d'u'$$
$$\sigma(u) = \sigma(d^{-1}d')\sigma(u') = \sigma(u')$$
$$u = u'$$
$$d = d'$$

So $D \cdot U$ is the union of disjoint open sets and for each $d \in D$ we have $dU$ for the connected component of $d$ in

$D \cdot U$ . Also, $\sigma$ is one-to-one on $dU$ whence it is a homeomorphism.

It follows, since $G$ is simply-connected, that $(\tilde{G},\sigma)$ is the trivial covering and that $\sigma$ is one-to-one (see Lecture XVIII). Hence $\sigma^{-1}$ is analytic (we are here using the fact that a one-to-one analytic homomorphism is a bianalytic isomorphism). Setting $\rho = \tau \circ \sigma^{-1} : G \longrightarrow H$ we verify easily that $\dot{\rho} = \theta$ .

Note 4: The commutator subgroup of the simply connected analytic group $G$ is closed.

Proof: Let $V$ be an analytic group whose Lie algebra is isomorphic to $\dot{G}/[\dot{G},\dot{G}]$ (e.g., $V = R^{\ell}$ where $\ell = \dim \dot{G} - \dim [\dot{G},\dot{G}]$) . Let $\theta$ be a homomorphism $\theta : \dot{G} \longrightarrow \dot{V}$ with $\ker \theta = [\dot{G},\dot{G}]$ .

There is a corresponding homomorphism $\rho : G \longrightarrow V$ with $\dot{\rho} = \theta$ . From the corollary to Note 2 we see that the analytic subgroup with Lie algebra $[\dot{G},\dot{G}]$ is closed and we have seen earlier (Lecture XIII) that this is the commutator subgroup.

Remark: Any real Lie algebra $\mathcal{H}$ is the Lie algebra of some analytic group $\mathcal{H}$ . The argument of Note 4 shows:

Any normal analytic subgroup $N$ of a simply-connected analytic group $G$ is closed.

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We now consider the case $G = H$ and in place of $\mathrm{Hom}(\dot{G},\dot{G})$ we look at $\mathrm{Aut}(\dot{G},\dot{G})$ . Note 1 shows that "·" maps injectively $\mathrm{Aut}(G,G)$ into $\mathrm{Aut}(\dot{G},\dot{G})$ and the mapping is surjective when $G$ is simply-connected. We will show now that $\mathrm{Aut}(\dot{G},\dot{G})$ is an

algebraic Lie group (in general it is true that $\mathrm{Aut}(G,G)$ is a Lie group but not necessarily algebraic).

Let $V$ be a finite-dimensional vector space over $R$. $\mathrm{Hom}_R(V,V)$ is isomorphic (as vector spaces) to $R^{(\dim V)^2}$. $\mathrm{Aut}_R(V,V)$ is an open submanifold of $\mathrm{Hom}_R(V,V)$ as it is the complement of the hypersurface defined by $\det x = 0$. $\mathrm{Aut}_R(V,V)$ is thus a Lie group with the two connected components $C_1 = \{x; \det x > 0\}$ and $C_2 = \{x; \det x < 0\}$.

Definition: A subgroup of $G(V) = \mathrm{Aut}_R(V,V)$ is called algebraic if it consists of all the common zeros in $G(V)$ of a set of functions in the ring generated by the linear functions on $\mathrm{Hom}_R(V,V)$ (this ring is called the ring of polynomial functions on $\mathrm{Hom}_R(V,V)$).

Remark: An algebraic subgroup of $G(V)$ is closed in the relative topology for $G(V)$. As the relative topology is the analytic group topology for $G(V)$ we conclude that:

> An algebraic subgroup, $A$, of $G(V)$ is closed in $G(V)$. Since $A$ is a closed subgroup of an analytic group it is a Lie group (Lecture XV).

We now look at $\mathrm{Aut}\ (\dot{G},\dot{G})$. An element $\sigma$ of $\mathrm{Aut}_R(\dot{G},\dot{G})$ is contained in $\mathrm{Aut}(G,\dot{G})$ just in case $[\sigma(X), \sigma(Y)] = \sigma[X,Y]$ for all $X,Y \in \dot{G}$. This condition is actually an algebraic one.

Lemma: The condition $[\sigma(X), \sigma(Y)] = \sigma[X,Y]$ is a condition that certain polynomials vanish on $\sigma$.

Proof: Let $X_1, X_2, \ldots, X_r$ be a basis for $\dot{G}$. We write:

$$\begin{cases} [X_i X_j] = \sum_k C^k_{ij} X_k \\ \sigma X_j = \sum_j \sigma_{ij} X_j \end{cases}$$

Then the condition $\sigma[X_i,X_j] = [\sigma(X_i),\sigma(X_j)]$ becomes,

$$\sum_{p,q} C^p_{ij}\, \sigma_{pq} X_q = [\sum_k \sigma_{ik} X_k \,,\, \sum_\ell \sigma_{j\ell} X_\ell] \,.$$

This, in turn, may be written

$$\sum_{p,q} C^p_{ij}\, \sigma_{pq} X_q = \sum_{k,\ell,q} \sigma_{ik}\, \sigma_{j\ell}\, C^q_{k\ell}\, X_q \,.$$

from which we obtain the polynomial conditions on the $\{\sigma_{ij}\}$ by equating components.

Thus $\mathrm{Aut}(\dot{G},\dot{G})$ is an algebraic group and, by virtue of the above remark, it is a Lie group.

Exercises: Making the standard identification of the Lie algebra of $G(V)$ with a subset of $\mathrm{Hom}_R(V,V)$ prove that the Lie algebra of $\mathrm{Aut}\ (\dot{G},\dot{G})$ is $\mathrm{Der}(\dot{G})$ (i.e.; the algebra of derivations of $\dot{G}$) .

We now take up the theory of covering groups in order to relate information about simply-connected groups to arbitrary groups.

Definition: A map $f:A \longrightarrow B$ is called an even covering of the space $B$ by the space $A$ if and only if each point $p \in B$ has a neighborhood $U_p$ such that each connected component of $f^{-1}(U_p)$ is mapped topologically onto $U_p$ by $f$. The pair $(A,f)$ is called a covering space of $B$.

Recall the following facts:

1) Given $M$ simply-connected and $f:A \longrightarrow B$ an even covering and $\varphi:M \longrightarrow B$ continuous, there is a continuous map $\tilde{\varphi}:M \longrightarrow A$ such that $\varphi = f \circ \tilde{\varphi}$

$$\begin{tikzcd}
M \arrow[r, dashed, "\tilde{\varphi}"] \arrow[dr, "\varphi"'] & A \arrow[d, "f"] \\
 & B
\end{tikzcd}$$

Moreover, given any $a_0 \in f^{-1}(\varphi(m_0))$, $m_0 \in M$, there is one and only one map $\tilde{\varphi}$ such that $\tilde{\varphi}(m_0) = a_0$. It follows that a simply-connected space has no non-trivial even covering.

2) Given a topological space $A$ which is locally connected and locally simply connected, it has a simply connected covering space $\tilde{A}$, which is unique up to a homeomorphism.

Definition: The Poincaré group of $A$ is the group of homeomorphisms $\gamma : \tilde{A} \longrightarrow \tilde{A}$ of the simply connected covering space $\tilde{A}$ such that the following diagram is commutative:

$$\begin{tikzcd} \widetilde{A} \arrow[rr, "\gamma"] \arrow[dr, "f"'] & & \widetilde{A} \arrow[dl, "f"] \\ & A & \end{tikzcd} \qquad f \circ \gamma = f$$

We denote it by $\pi_1(A)$ . From 1) it follows that $\widetilde{A}/\pi_1(A) \approx A$ , for given any points $\widetilde{a}_1$ and $\widetilde{a}_2$ both of which are in $f^{-1}(a)$ , $a \varepsilon A$ , there is exactly one element $\gamma$ in $\pi_1(A)$ such that $\gamma\widetilde{a}_1 = \widetilde{a}_2$

3) $\pi_1(A \times B) = \pi_1(A) \times \pi_1(B)$

4) Let $G$ be topological group. Let $(\widetilde{G},f)$ be a simply connected covering space. The $\widetilde{G}$ can be made into a topological covering space. The $G$ can be made into a topological group so that $f$ is a homomorphism:

Proof: $\pi_1(\widetilde{G} \times \widetilde{G}) = (1) \times (1) = (1)$ and therefore $\widetilde{G} \times \widetilde{G}$ is simply connected. Let $\mu\colon G \times G \longrightarrow G$ be defined by $\mu(x,y) = xy^{-1}$ . We have:

$$\begin{tikzcd} \widetilde{G} \times \widetilde{G} \arrow[r, dashed, "\widetilde{\mu}"] \arrow[d, "f \times f"'] & \widetilde{G} \arrow[d, "f"] \\ G \times G \arrow[r, "\mu"'] & G \end{tikzcd}$$

there exists $\widetilde{\mu}\colon \widetilde{G} \times \widetilde{G} \longrightarrow \widetilde{G}$ , a lift of $\mu \circ (f \times f)$ with the following restrictions:

a) Choose $\widetilde{1} \varepsilon f^{-1}(1)$

b) Choose $\widetilde{\mu}$ so that $\widetilde{\mu}(\widetilde{1},\widetilde{1}) = \widetilde{1}$

Given $\widetilde{x} \varepsilon \widetilde{G}$ set $\widetilde{x}^{-1} = \widetilde{\mu}(1,\widetilde{x})$ . Define $\widetilde{x} \cdot \widetilde{y} = \widetilde{\mu}(\widetilde{x},\widetilde{y}^{-1})$ .

Claim: This multiplication in $\tilde{G}$ is associative

Proof: $\tilde{x}(\tilde{y}\tilde{z}) = (\tilde{x}\tilde{y})\tilde{z}$ for each of these expressions represents a map of $\tilde{G} \times \tilde{G} \times \tilde{G} \longrightarrow \tilde{G}$ which covers the diagonal of the commutative diagram --

$$\begin{tikzcd}
\tilde{G} \times \tilde{G} \times \tilde{G} \arrow[r] \arrow[d, "(f \times f \times f)"'] & \tilde{G} \arrow[d, "f"] \\
G \times G \times G \arrow[r] & G \\
(x,y,z) \arrow[r] \arrow[u] & xyz \arrow[u]
\end{tikzcd}$$

As each takes $(\tilde{1},\tilde{1},\tilde{1})$ into $\tilde{1}$ they coincide. Also $\tilde{x}^{-1}\cdot\tilde{x} = \tilde{x}\cdot\tilde{x}^{-1} = \tilde{1}$. $\tilde{G}$ is a group and $f$ is a continuous homomorphism.

Remark: $\pi_1(G)$ can be identified with $\operatorname{Ker} f = f^{-1}(1)$ as a point set. This correspondence is actually an isomorphism of groups --

Proof: Let $\theta : \pi_1(G) \longrightarrow \operatorname{Ker} f$ be defined by $\gamma \in \pi_1(G) \longrightarrow \gamma(\tilde{1})$

- $\theta$ is bijective
- $(\gamma_2 \circ \gamma_1)(\tilde{1}) = \gamma_2(\tilde{1}) \cdot \gamma_1(\tilde{1})$

Thus, we can identify $G$ with $\tilde{G}/\operatorname{Ker} f$.

If $\tilde{G}$ is connected the discrete normal subgroup $D$ is contained in the center.

Proof: For $d \in D$ consider the map $g \longrightarrow gdg^{-1}$ which is a continuous map of a connected group to a discrete one. It follows that $gdg^{-1} = d$ for all $g \in G$ and $D$ is contained in the center of $G$.

5) In case $G$ is an analytic group we can give $\tilde{G}$ an analytic structure so that $\tilde{G}$ is an analytic group and

$f$ an analytic homomorphism. Thus every analytic group is covered by a simply-connected analytic group.

We have seen that the group of automorphisms of a simply-connected analytic group $\tilde{G}$, $\mathrm{Aut}(\tilde{G},\tilde{G})$ is equal to $\mathrm{Aut}(\mathfrak{g},\mathfrak{g})$ where $\mathfrak{g}$ is the Lie algebra of $\tilde{G}$. If $G$ is an analytic group whose Lie algebra is isomorphic to $\mathfrak{g}$ we can say that $G$ must be $\tilde{G}/D$ where $D$ is a discrete normal subgroup of $\tilde{G}$. $\mathrm{Aut}(G,G)$ can be identified with the subset of $\mathrm{Aut}(\tilde{G},\tilde{G})$ which keeps $D$ invariant (a Lie group since it is closed).

Exercise: Let $G$ be an analytic group. Let $H$ be an analytic subgroup such that $\overline{H} = G$ (e.g., a Kronecker line on the torus).

Then $[G,G] = [H,H]$

Definition: Given $N,G,A$ Lie groups with the following split exact sequence of analytic homomorphisms

$$\begin{tikzcd}(1) \arrow[r] & N \arrow[r, "\beta"] & G \arrow[r, shift left, "\alpha"] & A \arrow[l, shift left, "\alpha'"] \arrow[r] & (1)\end{tikzcd}$$

(i.e., $\alpha' : A \longrightarrow G$ and $\alpha \circ \alpha' =$ identity on $A$) we call $G$ a trivial extension of $N$ by $A$.

We have an analgous definition for Lie algebras

$$\left(\begin{tikzcd}(0) \arrow[r] & \mathfrak{n} \arrow[r, "\beta"] & \mathfrak{g} \arrow[r, shift left, "\alpha"] & \mathfrak{a} \arrow[l, shift left, "\alpha'"] \arrow[r] & (0)\end{tikzcd}\right).$$

As $\alpha'$ must be a momomorphism we can identify $A$ with an analytic subgroup (for Lie algebras $\alpha'(\mathfrak{a})$ need not be an ideal but we have $\mathfrak{g} = \mathfrak{n} \oplus \alpha'(\mathfrak{a})$ (semi-direct).

Exercise: Given $G$ a simply connected analytic group and given $\dot{G}$ is a trivial extension of $\dot{N}$ by $\dot{A}$, then $G$ is a trivial extension of $N$ by $A$.

Hint: Define analytic groups $N$ and $A$ and given an analytic homomorphism $\varphi : A \longrightarrow \mathrm{Aut}(N)$ we define $A \underset{\varphi}{\times} N$ (semi-direct product with respect to $\varphi$) as follows:

$$(a_1,n_1)(a_2,n_2) = (a_1a_2, \varphi(a_2)n_1 \cdot n_2) .$$

In this lecture we discuss some of the properties of solvable Lie groups.

Definition: A Lie group (algebra) is called solvable if and only if there exist subgroups (subalgebras) $G = G_0 \supset G_1 \supset \dots \supset G_n = (1)$ such that $G_{i+1}$ is normal in $G_i$ and $G_i/G_{i+1}$ is abelian. Equivalently, we have $G^{(n)} = (1)$ ((0) in the case of the algebra) for some integer $n$ where $G^{(i+1)} = [G^{(i)}, G^{(i)}]$ and $G^{(0)} = G$.

Thus, an analytic group is solvable if and only if its Lie algebra is solvable.

Lemma: Let $G$ be a topological group. Let $H$ be a normal subgroup. Then its topological closure $\overline{H}$ is a normal subgroup.

Proof: In as much as the automorphism $\Gamma_x : g \longrightarrow x^{-1}gx^{-1}$ is also a homeomorphism of $G$ the condition $x^{-1}Hx \subset H$ implies $x^{-1}\overline{H}x \subset \overline{H}$.

Lemma: Let $G$ be a topological group and let $H$ be a solvable subgroup. Then $\overline{H}$ is solvable.

Proof: Suppose we have $H \supset H^{(1)} \supset \dots \supset H^{(n)} = (1)$. Then also $\overline{H} \supset \overline{H}^{(1)} \supset \dots \supset \overline{H}^{(n)} = (1)$. It suffices to show $\overline{H^{(i)}}/\overline{H^{(i+1)}}$ is abelian. As $H^{(i)}/H^{(i+1)}$ is abelian we have $H^{(i)} \cdot \overline{H^{(i+1)}}/\overline{H^{(i+1)}}$ abelian and its closure (which must also be abelian) is $\overline{H^{(i)}} / \overline{H^{(i+1)}}$.

Lemma: In a Lie algebra $\mathfrak{g}$ the union of all solvable ideals is a solvable ideal.

Proof: Given $\mathcal{H}_1$ and $\mathcal{H}_2$ solvable ideals, the sum $\mathcal{H}_1 + \mathcal{H}_2$ is an ideal. Also if $\mathcal{H}_1^{(n_1)} = (0)$ and $\mathcal{H}_2^{(n_2)} = (0)$ with $n_1 \geq n_2$ we have $(\mathcal{H}_1 + \mathcal{H}_2)^{(n_1)} = (0)$.

As the union of all solvable ideals can be written as a finite sum of solvable ideals we see that it is a solvable ideal.

Definition: The union of all solvable ideals of $\mathcal{G}$ is called the radical of $\mathcal{G}$. It is the unique maximal solvable ideal of $\mathcal{G}$. Given an analytic group $G$ we define $R = \text{rad } G$ to be the analytic subgroup whose Lie algebra is $\text{rad } \dot{G}$ (the radical of $\dot{G}$).

Proposition: The radical $R$ of an analytic group $G$ is closed and is the maximum normal solvable connected subgroup.

Proof: $\bar{R}$ is analytic, normal, solvable, whence $\dot{\bar{R}}$ is a solvable ideal and $\dot{\bar{R}} = \dot{R}$. So $\bar{R} = R$ and $R$ is closed, normal, and solvable. That it is maximal among connected analytic subgroups with these properties results, also, from the maximality of $\dot{R}$.

Remark: There is a striking analogy with associative algebras. The very important Levi decomposition theorem, which historically was proved before the Wedderburn analogue for associative algebras asserts that for a simply connected analytic group $G$: The sequence $(1) \to R \to G \to G/R \to (1)$ splits. A proof of this theorem will be presented much later.

Proposition: The analytic group $G/R$ has no radical.

Proof: The radical, $H/R$, in $G/R$ gives $H \subset G$, a normal solvable, connected subgroup. Since $H \supset R$ we have $H = R$ and $H/R = (1) \in G/R$.

Remark: Originally the property of having no radical was taken as a definition of "semisimplicity". We usually use a different definition nowadays but one which is equivalent to this property. We will discuss the equivalence completely later.

Theorem 1: Let $G$ be a simply connected solvable analytic group. Then $G$ is homeomorphic to euclidean space. More precisely, $G$ can be expressed as a succession of semi-direct products of groups isomorphic to $R^1$.

Proof: $[G,G]$ is closed (Note 4, Lecture XVI). $G/[G,G]$ is an abelian analytic group and, since the map $G \longrightarrow G/[G,G]$ is monotone (implying that the induced map $\pi_1(G) \longrightarrow \pi_1(G/[G,G])$ is onto), $\pi_1(G/[G,G])$ is $(1)$. Thus $G/[G,G]$ is a simply connected, analytic, abelian group. It is therefore isomorphic to $R^k$ (where $k = \dim G/[G,G]$), whence $G/[G,G] \approx R_1 \times R_2 \times \dots \times R_k$ (direct) where $R_i \approx R^1$.

Let $G_1$ = inverse image of $R_1$

$G_2$ = inverse image of $R_1 \times R_2$

$\vdots$

$G_\ell$ = inverse image of $R_1 \times R_2 \times \dots \times R_\ell$.

We have $G = G_\ell \supset G_{\ell-1} \supset \cdots \supset [G,G]$ and each $G_i$ is a normal analytic subgroup (and hence it is closed -- note 4, Lecture XVI). Now $G = R_\ell \times G_{\ell-1}$ (semi-direct) because $\dot{G} = \dot{R}_\ell + \dot{G}_{\ell-1}$ and $\dot{R}_\ell \cap \dot{G}_{\ell-1} = (0)$. Thus $\pi_1(G) = \pi_1(R_\ell) \times \pi_1(G_{\ell-1})$ and $\pi_1(G_{\ell-1})$ is therefore (1) so that $G_{\ell-1}$ is simply connected and we may proceed by induction.

So for simply connected analytic groups the radical contributes only a euclidean space (topologically).

---

We now come to the very important Lie's theorem for solvable groups. Let $V$ be a finite dimensional (non-zero) vector space over the field $K$. As before we denote by $G(V)$ the group of non-singular transformations of $V$. A subgroup (not necessarily algebraic) of $G(V)$ is called a linear group. When $K$ is the field of complex numbers $G(V)$ has the relative topology of the euclidean space $\mathrm{Hom}_K(V,V)$,

Theorem 2: Let $G$ be a solvable, connected subgroup of $G(V)$. Then there is a basis for $V$ with respect to which the transformations $g \in G$ all have triangular matrices.

This theorem is equivalent to:

Theorem 2': Let $G$ be a solvable, connected subgroup of $G(V)$. Then there is a sequence of subspaces --

$$V = V_0 \supset V_1 \supset \cdots \supset V_n = (0)$$

such that:

1) $gV_i = V_i$ for all $g \in G$

2) $\dim V_i = \dim V_{i+1} + 1$

Thus to prove this theorem it suffices to prove that there is a non-zero element $w \in V$ such that $g(w) = \lambda(g)w$ for each $g \in G$. For given this, take $V_{n-1} = Kw$ and apply this fact to $V/V_{n-1}$. We may thus state the following equivalent formulation of Theorem 2:

Theorem 2": Let $G$ be a connected solvable linear group on $V$. Then there is in $V$ a non-zero element $v_0 \in V$ such that $v_0$ is a common eigenvector for all transformations of $G$; i.e., $gv_0 = \lambda(g)v_0$ where $\lambda(g) \in K$.

There is also such a theorem for solvable linear Lie algebras over an algebraically closed field of characteristic $0$. We give the proof here for groups because (with slight modifications) it goes over to arbitrary characteristic (in characteristic $p > 0$ we mean by "connected" that $G$ should be connected in the Zariski topology). In the last 10 years -- since the proof of Theorem 2" (by Kolchin) -- we find a tendency to do things on the group directly for all characteristics simultaneously (especially in case $G$ is compact).

Proof of Theorem 2": $G$ is connected and solvable. Hence we have:

$$G \supset G^{(2)} = [G,G] \supset \ldots \supset G^{(n-1)} \supset G^{(n)} = (1)$$

and

1) each $G^{k} = [G^{(k-1)}, G^{(k-1)}]$ is connected (the commutator subgroup of a connected group is connected).

2) $G^{(n-1)}$ is abelian.

We begin by studying $G^{(n-1)}$ and build the proof from there.

Definition: Let $\mu : H \longrightarrow K^*$ be a homomorphism of the linear group $H$ into the multiplicative group of $K$. . We set

$$\bar{V}_\mu(H) = \{v;\ v \in V \text{ and } h(v) = \mu(h)v \text{ for all } h \in H\}$$

Take $g \in G^{(n-1)}$ and let $\lambda$ be any one of its eigenvalues. Let $\bar{V}_\lambda(g) = \{v; v \in V \text{ and } gv = \lambda v\}$ . By construction $\bar{V}_\lambda(g)$ is not $(0)$ and, furthermore, $\bar{V}_\lambda(g)$ is invariant under $G^{(n-1)}$ since, for $g' \in G^{(n-1)}$ we have:

$$g(g'v) = g'gv = g'(\lambda v) = \lambda(g'v) \quad .$$

We now consider $G^{(n-1)}$ operating on $\bar{V}_\lambda(g)$ . If $G \neq G^{(n-1)}$ or $\bar{V}_\lambda(g) \neq V$ we can apply induction to find a non-zero element $v_1 \in \bar{V}_\lambda(g)$ such that $gv_1 = \nu(g)v_1$ for all $g \in G^{(n-1)}$ .

For each homomorphism $\mu : G^{(n-1)} \longrightarrow K^*$ set $\bar{V}_\mu = \bar{V}_\mu(G^{(n-1)})$ . Let us assume, for convenience, that we have here the case $G = G^{(0)} \supset G^{(1)} = G^{(n-1)} \supset G^{(2)} = (1)$ . In the general case, we must actually perform a double induction on $n$ and the dimension of $V$ . This amounts to an easy modification of what follows. For $\nu : G^{(1)} \Rightarrow K^*$ and $g \in G$ we define $\nu' = \sigma(g)\nu : G^{(1)} \Rightarrow K^*$ by $\nu'(g') = \nu(g^{-1}g'g)$ for $g' \in G^{(n-1)}$ . Then we have $g\bar{V}_\nu = \bar{V}_{\nu' = \sigma(g)\nu}$ :

Proof: $g'gv = g(g^{-1}g'g)v$ for $g' \in G^{(n-1)}$, $v \in V_\nu$ , $g \in G$ .

$$= g\nu(g^{-1}g'g)v$$

$$= \nu(g^{-1}g'g)gv$$

That is, $gv \in \bar{V}_{\nu'}$

Now there are only a finite number of homomorphisms $\mu$ involved ($\overline{V}_\mu = 0$ for almost all $\mu$). Hence the map $\sigma$ which takes $g \longrightarrow \sigma(g)$ is a map from a connected group to a finite one and hence it is constant. Thus $G\overline{V}_\nu = \overline{V}_\nu$ and if $\overline{V}_\nu \neq V$ we are finished by induction. So we may assume $\overline{V}_\nu = V$; i.e.,

$$gv = \nu(g)v \quad \text{for all} \quad g \in G^{(1)} \quad \text{and} \quad v \in V .$$

It results that $G^{(1)}$ acts on $V$ by stretchings and therefore, lies in the center of $G$.

Lemma: For $g \in G$ we have:

$$G(\overline{V}_\lambda(g)) \subset \overline{V}_\lambda(g)$$

Proof: Set $H$ = subgroup generated by $g$ and $G^{(1)}$. Since $G/G^{(1)}$ is abelian, $H$ is normal in $G$. Also,

$\overline{V}_\lambda(g) = \overline{V}_\mu(H)$ where $\mu$ is a homomorphism $\mu : H \longrightarrow K^*$ with $\mu(g) = \lambda$.

Now $G(\overline{V}_\mu(H)) \subset \overline{V}_\mu(H)$ since $G$ is connected.

Thus if for some $g$ we have $\overline{V}_\lambda(g) \neq V$ we are finished by induction. We are, therefore, reduced to the case $\overline{V}_\lambda(g) = V$ for all $g$ and all $\lambda$ such that $\overline{V}_\lambda(g) \neq 0$. Now for any $g$ there is a $\lambda$ such that $\overline{V}_\lambda(g) \neq 0$ hence $\overline{V}_\lambda(g) = V$. So any non-zero vector $v \in V$ will do.

Corollary: Let $\mathcal{G}$ be a solvable Lie subalgebra of linear endomorphisms of a finite dimensional (non-zero) vector space $V$ over the field $K$ of complex numbers. Then there exists a non-zero $v \in V$ such that $g\,v = \lambda(g)v$ for any $g \in \mathcal{G}$ ($\lambda : \mathcal{G} \longrightarrow K$ is a homomorphism of $\mathcal{G}$ into the additive group of $K$.

# LECTURE XIX

We now begin the representation theory of Lie algebras.

Let $T$ be an endomorphism of the linear space $V$ over the field $K$.

Definition: $T$ is called nilpotent if and only if there is a positive integer $n$ such that $T^n = 0$.

$T$ is called semi-simple when its minimal polynomial has no repeated factors.

We make frequent use, in the following lectures, of this fact from linear algebra (for a proof see Chevalley, Vol. II, Groupes Algebriques):

- Given any $T$ there exist a unique semi-simple endomorphism $S$ and a unique nilpotent endomorphism $N$ such that:

$$1) \quad T = S + N$$

$$2) \quad SN = NS$$

In case $K$ is algebraically closed or, more generally, if the roots of the minimal polynomial of $T$ all lie in $K$, this fact corresponds to the Jordan normal form.

Let $V^* = \mathrm{Hom}_K(V,K)$ be the dual space of $V$. We define the space $V_{r,s}$ by

$$V_{r,s} = \underbrace{V \otimes V \otimes \dots \otimes V}_{r \text{ copies}} \otimes \underbrace{V^* \otimes V^* \otimes \dots \otimes V^*}_{s \text{ copies}}$$

Given an automorphism $g$ of $V$ we define an automorphism $g^*$ of $V^*$ by $g^* = {}^t g^{-1}$; i.e.,

$$\lambda(v) = < v,\lambda > = < gv, g^*\lambda >$$

for all $v \in V$ and $\lambda \in V^*$. We define $g_{r,s} \in Gl(V_{r,s})$ by

$$g_{r,s} = \underbrace{g \otimes g \otimes \cdots \otimes g}_{r \text{ copies}} \otimes \underbrace{g^* \otimes g^* \otimes \cdots \otimes g^*}_{s \text{ copies}}$$

obtaining the mapping $\theta_{r,s} : Gl(V) \longrightarrow Gl(V_{r,s})$. The homomorphism $\theta_{r,s}$ is analytic in case $K$ is the real field $R$. Keeping in mind that we have $\dot{Gl}(V)$ (the Lie algebra of $Gl(V)$) identified with $\mathcal{E}(V) = \mathrm{Hom}_K(V,V)$ we define the map $\dot\theta_{r,s} : \dot{Gl}(V) \longrightarrow \dot{Gl}(V_{r,s})$ as follows:

Definition: Given $T \in \mathrm{Hom}_K(V,V)$ we define

$$\begin{aligned}(*) \quad \dot\theta_{r,s}(T) = \; & T \otimes I \otimes \cdots \otimes I \otimes I^* \otimes I^* \otimes \cdots \otimes I^* \\ & + I \otimes T \otimes \cdots \otimes I \otimes I^* \otimes I^* \otimes \cdots \otimes I^* \\ & \;\; \vdots \\ & \; I \otimes I \otimes \cdots \otimes I \otimes T \otimes I^* \otimes I^* \otimes \cdots \otimes I^* \\ & + I \otimes I \otimes \cdots \otimes I \otimes (-{}^tT) \otimes I^* + \cdots + I^* \\ & + \;\vdots \\ & \; I \otimes I \otimes \cdots \otimes I \otimes I^* \otimes I^* \otimes \cdots \otimes I^* \otimes (-{}^tT)\end{aligned}$$

so that we have the following situation:

Set $\otimes V = \sum_{r,s} V_{r,s}$ (direct) and $\theta = \sum_{r,s} \theta_{r,s}$.

$\theta(g)$ is an automorphism of $\otimes V$ and $\dot\theta(g) = \Sigma \dot\theta_{r,s}(g\delta$ is a derivation of $\otimes V$.

Note that the natural identification of $V_{1,1} = V \otimes V^*$ with $\mathrm{Hom}(V,V)$, in which $v \otimes \lambda$ operates on $w \in V$ by

$$(v \otimes \lambda)w = \lambda(w)v ,$$

is compatible with the operations of $Gl(V)$ and $\dot{Gl}(V)$. For $g \in Gl(V)$, $x \in Hom(V,V)$, $T \in \dot{Gl}(V) = Hom(V,V)$:

$$\theta_{1,1}(g)X = gXg^{-1}$$

$$\dot{\theta}_{1,1}(T)X = TX - XT = [T,X] .$$

Additional properties of $\dot{\theta}_{r,s}$ are:

1) $N$ nilpotent implies $\dot{\theta}_{r,s}(N)$ nilpotent, since (by virtue of (*)) $\dot{\theta}_{r,s}$ is a sum of nilpotent endomorphisms which commute.

2) $S$ semisimple implies $\dot{\theta}_{r,s}(S)$ is semi-simple, as we see by diagenalizing $S$ (Given an endomorphism $T$ of $V$, then upon extension of the ground field $T$ extends uniquely to an endomorphism of the enlarged space. Nilpotency and semi-simplicity are preserved under extension, since the minimal polynomial doesn't change. In the case of a perfect ground field an endomorphism is semi-simple if and only if its minimal polynomial has no repeated roots. Thus a semi-simple endomorphism is diagonalizable.

3) $\dot{\theta}_{r,s}$ is a Lie algebra homomorphism and, because $[\dot{\theta}_{r,s}(S), \dot{\theta}_{r,s}(N)] = \dot{\theta}_{r,s}([S,W]) = 0$ we have

$$\dot{\theta}_{r,s}(S) + \dot{\theta}_{r,s}(N)$$

is the decomposition of $\dot{\theta}_{r,s}(S+N)$ into semi-simple and nilpotent parts.

4) The eigenvalues of $T = S+N$ are the eigenvalues of $S$. It follows that the eigenvalues of $\dot{\theta}_{r,s}(T)$

are $\lambda_{i_1} + \dots + \lambda_{i_r} - (\lambda_{j_1} + \dots + \lambda_{j_s})$ where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $T$ (apply formula (*) to $S$ ).

0

Given $\mathfrak{H}$ a Lie subalgebra of $\mathrm{Hom}(V,V)$ , a tensor $t \in V_{r,s}$ is called an invariant of $\mathfrak{H}$ whenever $\dot{\theta}_{r,s}(\mathfrak{H})t = 0$ . This is equivalent to

$$\theta_{r,s}(H)t = t$$

where $H$ is the analytic group corresponding to $\mathfrak{H}$ (this also holds for algebraic groups and algebraic Lie algebras). The tensor $t \in V_{r,s}$ is called a semi-invariant of $\mathfrak{H}$ if $t$ is an eigenvector for $\dot{\theta}_{r,s}(\mathfrak{H})$ .

Given $W \subset V$ and $T \in \mathrm{Hom}(V,V)$ with $T(W) \subset W$ we set $Tw$ = restriction of $T$ to $W$ and $T_{V/W}$ = induced homomorphism of $V/W$ defined by the following diagram:

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d] & V \arrow[d] \\
V/W \arrow[r, "T_{V/W}"'] & V/W
\end{tikzcd}$$

If $T = S+N$ then (since $S$ and $N$ are polynomials in $T$) we have

$$Tw = S_w + N_w$$

$$T_{V/W} = S_{V/W} + N_{V/W}$$

Any tensor invariant of $T$ is a tensor invariant of its semisimple part $S$ and of its nilpotent part $N_s$ for if $T$ annihilates

a vector so does any polynomial in $T$ .

Given an endomorphism $T$ and a scalar $\lambda \varepsilon K$ we define $V_\lambda(T) = \{v;\ v \varepsilon V$ and $(T - \lambda I)^n v = 0$ for some $n\}$ . Then we easily check the following facts:

1) $V_\lambda(T) = \overline{V_\lambda(S)} = V_\lambda(S)$

2) If the minimal polynomial of $T$ has all its roots in the ground field $K$ , then

$$V = \sum_{\lambda \varepsilon K} V_\lambda(T) \qquad \text{(direct)}$$

We remark that in 2) the sum $\sum_{\lambda \varepsilon K} V_\lambda(T)$ is actually finite since $V_\lambda(T) = 0$ if $\lambda$ is not an eigenvalue of $T$ (this follows readily from 1)).

Lemma: Let $\mathcal{E} =$ Hom$(V,V)$ and $T \varepsilon \mathcal{E}$ . Let $a$ and $b$ be scalars. Then

$$\mathcal{E}_a(\text{ad } T)\ V_b(T) \subset V_{a+b}(T)$$

Proof: ad $T$ has for its decomposition into semi-simple and nilpotent parts

$$\text{ad } T = \text{ad } S + \text{ad } N \ .$$

$$\mathcal{E}_a(\text{ad } T) = \mathcal{E}_a(\text{ad } S) \ .$$

$$V_b(T) = V_b(S) \ .$$

Hence $\mathcal{E}_a(\text{ad } T)V_b(T) = \mathcal{E}_a(\text{ad } S)V_b(S)$ .

Now $X \varepsilon \mathcal{E}_a(\text{ad } S)$ just in case $[S,X] = aX$ and $v \varepsilon V_b(S)$ if and only if $sv = bv$ . We have :

$$\begin{aligned} SXv &= XSv + [S,X]v \\ &= X(bv) + aXv \\ &= (b+a)Xv \ . \end{aligned}$$

Thus $Xv$ belongs to the eigenvalue $(a+b)$ of $S$ .

0

We now give a proof of Lie's theorem for solvable linear Lie algebras over an algebraically closed field $K$ of characteristic = 0 .

Proof: Let $\mathfrak{g}$ be a solvable Lie algebra of endomorphisms of $V$ over the algebraically closed field $K$ of characteristic 0 . Let $\mathfrak{g}_1$ be an ideal of codimension = 1 . By induction we obtain an eigenvector $v$ common to all of $\mathfrak{g}_1$ . We may write:

$$\mathfrak{g} = KA + \mathfrak{g}_1 \quad \text{(as a linear space over } K\text{ )}$$

We have the space $W = \{v, Av, A^2v, \dots\}$ stable under $A$ . For $X \varepsilon \mathfrak{g}_1$ we have:

$$XAv = AXv + \underset{\substack{\shortparallel \\ X_1}}{[X,A]}v = \lambda(X)Av + \lambda(X_1)v$$

$$\begin{aligned} XA^2v &= AXAv + [X,A]Av = \lambda(X)A^2v + \lambda(X_1)Av + \lambda(X_1)Av \\ &\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \lambda([X_1,A])v \\ XA^kv &= \lambda(X)A^kv + \dots\dots \end{aligned}$$

That is, $W$ is invariant under $\mathfrak{g}_1$ and the matrix of $X$ with respect to the basis $\{v, Av, \dots, A^kv\}$ of $W$ is triangular and all eigenvalues are $\lambda(X)$ .

Thus $\operatorname{tr} X_W = (\dim W)\,\lambda(X)$ . If $X \varepsilon [\mathfrak{g},\mathfrak{g}]$ we have $X_W \varepsilon [\mathfrak{g}_W,\mathfrak{g}_W]$ and $\operatorname{tr} X_W = 0$ . Since char $K = 0$ the equation

$$(\dim W)\,\lambda(X) = 0$$

implies $\lambda(X) = 0$ for all $X \varepsilon [\mathfrak{g},\mathfrak{g}]$ . Then

$$XA^S v = \lambda(X) A^S v \quad \text{for} \quad X \varepsilon \mathfrak{g}_1$$

and $\mathfrak{g}_1$ operates on $W$ by stretching. So, take $w \varepsilon W$ an eigenvector for $A$ (possible since $K$ is algebraically closed) . Then $w$ is an eigenvector for $\mathfrak{g}$ .

We now develop some of the theory of Lie algebras. We begin with some facts about nilpotent Lie algebras.

Theorem (Engel): Let $\mathcal{G}$ be a Lie subalgebra of $\mathrm{Hom}_K(V,V)$ all of whose elements are nilpotent. Then there exists a non-zero element $v \varepsilon V$ such that $\mathcal{G}v = 0$.

Proof: We prove the theorem by induction on the dimension of $\mathcal{G}$. The theorem is certainly true if $\dim \mathcal{G} = 1$. Now assume $\mathcal{G}$ is of dimension greater than 1. Let $\mathcal{G}_1$ be a one dimensional subalgebra and consider $\mathrm{ad}\,\mathcal{G}_1$ operating in $\mathcal{G}$. We have:

$$\mathrm{ad}\,\mathcal{G}_1(\mathcal{G}_1) \subset \mathcal{G}_1$$

and we can consider as $\mathcal{G}_1$ operating in $\mathcal{G}/\mathcal{G}_1$. Applying induction (ad of a nilpotent endomorphism is nilpotent -- under the canonical tensor representation a nilpotent endomorphism is sent into a nilpotent endomorphism), we get a non-zero element $X_2 \not\in \mathcal{G}_1$ such that

$$\mathrm{ad}\,\mathcal{G}_1(X_2) \subset \mathcal{G}_1 .$$

Let $\mathcal{G}_1 + KX_2 = \mathcal{G}_2$. Then

$$[\mathcal{G}_1, \mathcal{G}_2] \subset \mathcal{G}_1 ;$$

i.e., $\mathcal{G}_1$, is an ideal in $\mathcal{G}_2$.

Applying induction to $\mathrm{ad}\,\mathcal{G}_2$ operating in $\mathcal{G}/\mathcal{G}_2$ we get an element $X_3 \not\in \mathcal{G}_2$ such that

$$\mathrm{ad}(\mathcal{G}_2)X_3 \subset \mathcal{G}_2$$

and $\mathcal{G}_3 = \mathcal{G}_2 + KX_3$ , whence $\mathcal{G}_2$ is an ideal in $\mathcal{G}_3$ .

Proceding in this way we obtain a series

$$\mathcal{G}_1 \subset \mathcal{G}_2 \subset \cdots \subset \mathcal{G}_{n-1} \subset \mathcal{G}_n = \mathcal{G}$$

with $\mathcal{G}_i$ an ideal of codimension 1 in $\mathcal{G}_{i+1}$ (it results immediately that $\mathcal{G}$ is solvable). Applying induction to $\mathcal{G}_{n-1}$ on V we get :

$$\bar{V}_0(\mathcal{G}_{n-1}) \neq 0$$

We can write $\mathcal{G} = KX + \mathcal{G}_{n-1}$ .

Take $v \neq 0$ such that $\mathcal{G}_{n-1}v = 0$ and let Y be any element of $\mathcal{G}_{n-1}$ . Then

$$YXv = XYv + [Y,X]v = 0$$

Thus $X(\bar{V}_0(\mathcal{G}_{n-1})) \subset \bar{V}_0(\mathcal{G}_{n-1})$ and X has a non-zero eigenvector there (in $\bar{V}_0(\mathcal{G}_{n-1})$) . Let $v_0$ be an eigenvector for X . Then:

$$\mathcal{G} v_0 = KXv_0 + \mathcal{G}_{n-1}v_0 = 0$$

Definition: A Lie algebra, $\mathcal{G}$ , is called nilpotent if and only if ad X is a nilpotent endomorphism of the underlying vector space of $\mathcal{G}$ for each $X \in \mathcal{G}$ .

Exercise: Let G be an analytic subgroup of the full linear group whose Lie algebra consists of nilpotent endomorphisms. Prove that G is simply connected (Hint: consider the logarithm) .

Definition: The normalizer $N(\mathfrak{H})$ of a subalgebra $\mathfrak{H}$ of $\mathfrak{G}$ is the maximal subalgebra in which $\mathfrak{H}$ is an ideal.

Definition: A Cartan subalgebra of a Lie algebra $\mathfrak{G}$ is a subalgebra $\mathfrak{H}$ satisfying:

$$\mathfrak{G}_0(\operatorname{ad} \mathfrak{H}) = \mathfrak{H}$$

Lemma: $\mathfrak{H}$ is a Cartan subalgebra if and only if:

1) $\mathfrak{H}$ is nilpotent

2) $N(\mathfrak{H}) = \mathfrak{H}$

Proof: $(\longrightarrow)$ Assume $\mathfrak{H}$ is a Cartan subalgebra; i.e., $\mathfrak{G}_0(\operatorname{ad} \mathfrak{H}) = \mathfrak{H}$. Then 1) $\mathfrak{H}$ is certainly nilpotent as for any $h, h' \varepsilon \mathfrak{H}$ we have

$$(\operatorname{ad} h)^n h' = 0$$

2) $[\mathfrak{H}, N(\mathfrak{H})] \subset \mathfrak{H}$ implies $N(\mathfrak{H}) \subset \mathfrak{G}_0(\operatorname{ad} \mathfrak{H}) = \mathfrak{H}$.

$(\longleftarrow)$ Assume $\mathfrak{H}$ is a nilpotent subalgebra equal to its own normalizer. Then $\mathfrak{H}$ nilpotent implies $\mathfrak{H} \subset \mathfrak{G}_0(\operatorname{ad} \mathfrak{H})$. From $N(\mathfrak{H}) = \mathfrak{H}$ we see that $(\operatorname{ad} \mathfrak{H})_{\mathfrak{G}/\mathfrak{H}}$ has no non-zero eigenvector corresponding to eigenvalue $0$. We have, by Engel's theorem, $(\mathfrak{G}/\mathfrak{H})_0\ (\operatorname{ad} \mathfrak{H}_{\mathfrak{G}/\mathfrak{H}}) = 0$. So

$$\mathfrak{G}_0(\operatorname{ad} \mathfrak{H})/\mathfrak{H} = 0$$

Remarks: 1) A Cartan subalgebra is an maximal nilpotent subalgebra but not conversely; e.g., in $\operatorname{Hom}(V,V)$ the super-diagonal matrices form a maximal nilpotent subalgebra but not a Cartan subalgebra.

2) A nilpotent Lie algebra $\mathfrak{G}$ is hypercentral and conversely; i.e., there exists a series $\mathfrak{G} = \mathfrak{G}_0 \supset \mathfrak{G}_1 \supset \mathfrak{G}_2 \supset \cdots \supset \mathfrak{G}_n = \{0\}$ such that:

1) $\mathcal{G}_i$ is an ideal in

2) $\mathcal{G}_i/\mathcal{G}_{i+1}$ is the center of $\mathcal{G}/\mathcal{G}_{i+1}$ .

This may be seen by first proving that $\mathcal{G}$ is nilpotent if and only if $\mathcal{G}_k = 0$ for some $k$ where we define the sequence of $\mathcal{G}$-ideals $\mathcal{G}_1 = \mathcal{G}$ and $\mathcal{G}_k = [\mathcal{G}_{k-1}, \mathcal{G}]$ . (Apply Engel's theorem to the adjoint group).

Exercise: The derived subalgebra, $[\mathcal{G},\mathcal{G}]$ , of a solvable subalgebra $\mathcal{G}$ is nilpotent over a field $K$ of characteristic $0$ . (Hint:Apply Lie's theorem in the linear case.)

Definition: The Killing polynomial of the Lie algebra $\mathcal{G}$ is defined by:

$P(\lambda,X) = \det(\lambda I - \mathrm{ad}X)$ where $\lambda$ is an

indeterminant and $X$ is an indeterminant element of $\mathcal{G}$ ($\lambda$ and $X$ are independent variables). This is a polynomial

$$\lambda^n - B_1(X)\lambda^{n-1} + B_2(X)\lambda^{n-2} + \dots + B_k(X)\lambda^{n-k} .$$

Each $B_k(X)$ is a polynomial function on $\mathcal{G}$ ; i.e., in the ring generated by the linear functions on $\mathcal{G}$ (the latter ring is graded and $B_k$ is homogeneous of degree $k$).

. Each of the polynomials $B_k$ is invariant under ad $\mathcal{G}$ -- this means (over the real or complex numbers)

$$B_k(gX) = B_k(X) \quad \text{for all } g \text{ in the "adjoint group".}$$

We now make this precise and prove the invariance of the $B_k$'s .

Definition: The adjoint group of a Lie algebra $\mathfrak{g}$ over the real numbers may be defined as the analytic group of automorphisms determined by the derivations of $\mathfrak{g}$; i.e., the analytic group corresponding to the linear Lie algebra of derivations of $\mathfrak{g}$. Alternately, it may be defined as $\text{Ad}\, G = \{\text{differentials of automorphisms of } G\}$ where $G$ is any analytic group whose Lie algebra is $\mathfrak{g}$.

Proposition: Over an arbitrary ground field $K$ of characteristic $0$, the polynomials $B_k(X)$ are invariant. That is $B_k(X_t)$ is constant where

$$\begin{aligned} X_t &= X + t[Y,X] \\ &= (I + t\,\text{ad}\, Y)(X) \end{aligned}$$

Proof: We want to show

$$\left.\frac{d}{dt}\right|_{t=0} B_k(X_t) = 0 \quad \text{for each } k .$$

That is, $\left.\frac{d}{dt}\right|_{t=0} P(\lambda,X_t) = 0$ (where the differentiation is formal). In other words, we want to show that the coefficient of $t$ in $P(\lambda,X_t) - P(\lambda,X)$ is zero.

$$\begin{aligned} \text{Now}\quad P(\lambda,X_t) &= \det(\lambda I - \text{ad}\, X_t) \\ &= \det(\lambda I - \text{ad}\, X - \text{ad}\, t[Y,X]) \\ &= \det(\lambda I - \text{ad}\, X)\det(I - t(\lambda I - \text{ad}X)^{-1}\text{ad}[Y,X]) \end{aligned}$$

We may write the coefficient of $t$

$$\det(\lambda I - \text{ad}X)(1 - t\, \text{Tr}(\lambda I - \text{ad}X)^{-1}\text{ad}[Y,X]) + t^2 m$$

since only the diagonal terms contribute.

Coefficient of $t$ is then:

$$\det(\lambda I-\mathrm{ad}X)[\mathrm{Tr}\,\alpha(\mathrm{ad}Y\mathrm{ad}X - \mathrm{ad}X\mathrm{ad}Y)]$$

$$(*) \quad = \det(\lambda I-\mathrm{ad}X)\cdot(\mathrm{tr}(\alpha\mathrm{ad}Y)\mathrm{ad}X-\mathrm{Tr}\,\mathrm{ad}X(\alpha\mathrm{ad}Y)$$

$$= 0$$

where $\alpha = (\lambda I-\mathrm{ad}X)^{-1}$ .

Note 1: This argument goes through if we write $X_t = [I+tD](X)$ where $D$ is a derivation of $\mathcal{G}$ . The only change is that we get trace $\alpha$ ad $D(X)$ at $(*)$ and then since ad $D(X) = [\mathrm{ad}X,D] = (\mathrm{ad}X)D - D(\mathrm{ad}X)$, again trace $\alpha\, D(X) = 0$ .

Note 2: From the invariance of the $B_k$'s we get:

1) The zeros of $B_1$ form an ideal

2) $B_n(X) = 0$ for all $X$ in $\mathcal{G}$ .

Definition: The rank of $\mathcal{G}$ = supremum of all $r$ such that $\lambda^r | P(\lambda,X)$ .

# Lecture XXI

Let $V$ be a vector space over the ground field $K$. Let $T$ be an endomorphism of $V$, and $a \in K$. As before we denote $\mathcal{E} = \operatorname{Hom}(V,V)$, and $V_0(T) = \{v;\ v \in V;\ Tv = 0\}$. For a set $\mathcal{S}$ of endomorphisms of $V$ we write $V_0(\mathcal{S}) = \bigcap_{T \in \mathcal{S}} V_0(T)$. For any $a,b$ in $K$, $T \in \mathcal{E}$ we have shown

$$\mathcal{E}_a(\operatorname{ad} T) V_b(T) \subset V_{a+b}(T).$$

**Proposition** $[\mathcal{G}_a(\operatorname{ad} X), \mathcal{G}_b(\operatorname{ad} X)] \subset \mathcal{G}_{a+b}(\operatorname{ad} X)$

**Proof** $[\operatorname{ad} X, \operatorname{ad} Y] = \operatorname{ad}[X,Y]$ Set $V = \mathcal{G}$.

Denote by $\hat{\operatorname{ad}}$ the adjoint representation of the Lie algebra $\mathcal{E}$. Then $(\hat{\operatorname{ad}}\ \operatorname{ad}(X))(\operatorname{ad} Y) = \operatorname{ad}(\operatorname{ad} X(Y))$. That is

$$\hat{\operatorname{ad}}\ u \circ \operatorname{ad} = \operatorname{ad} \circ u \quad \text{for all} \quad u \in \operatorname{ad} \mathcal{G}.$$

Hence

$$(\hat{\operatorname{ad}}\ u - a)^p \operatorname{ad} Y = \operatorname{ad} \circ (u - a)^p Y \quad \text{for}$$

all $u \in \operatorname{ad} \mathcal{G}$ and $Y \in \mathcal{G}$.

It follows at once that

$$\operatorname{ad} \mathcal{G}_a(u) \subset \mathcal{E}_a(\hat{\operatorname{ad}}\ u)$$

for all $u \in \operatorname{ad} \mathcal{G}$. The proposition now follows from $\mathcal{E}_a(\hat{\operatorname{ad}}\ u)\, \mathcal{G}_b(u) \subset \mathcal{G}_{a+b}(u)$.

Thus, in particular, $\mathcal{G}_0(\operatorname{ad} X)$ is a subalgebra for any $X$ in $G$.

**Note**: The rank of $\mathcal{G}$ may alternately be defined as the minimum dimension of $\mathcal{G}_0(\operatorname{ad} X)$ for all $X \in \mathcal{G}$.

**Definition**. $X \in \mathcal{G}$ is called regular if and only if $\dim \mathcal{G}_0(\operatorname{ad} X)$ is minimal, i.e., $X$ is regular in

case rank $(\mathcal{G}) = \dim \mathcal{G}_0(\mathrm{ad}\ X)$.

__Remark__. Let $\mathcal{F}$ be a subalgebra of $\mathcal{G}$, $X \in \mathcal{F}$. If $X$ is regular in $\mathcal{G}$ then $X$ is regular in $\mathcal{F}$.

__Proof__ $\dim \mathcal{G}_0(\mathrm{ad}\ X) = \dim \mathcal{F}_0(\mathrm{ad}\ X_{\mathcal{F}}) + \dim (\mathcal{G}/\mathcal{F})_0(\mathrm{ad}\ X_{\mathcal{G}/\mathcal{F}})$. The condition for regularity is that a certain polynomial does not vanish. We can choose $Y \in \mathcal{F}$ minimizing both $\dim \mathcal{F}_0(\mathrm{ad}\ Y)$ and $\dim (\mathcal{G}/\mathcal{F})_0(\mathrm{ad}\ Y)$ separately, for $Y$ is to be chosen so as not to annihilate a certain two polynomial functions on $\mathcal{F}$. Thus $\dim \mathcal{G}_0(\mathrm{ad}\ Y)$ $= \dim \mathcal{F}_0(\mathrm{ad}\ Y) + \dim (\mathcal{G}/\mathcal{F})_0(\mathrm{ad}\ Y) \leq \dim \mathcal{G}_0(\mathrm{ad}\ X)$. But $X$ is regular. Hence $\dim \mathcal{G}_0(\mathrm{ad}\ X) =$ $\dim \mathcal{G}_0(\mathrm{ad}\ Y)$. Therefore $\dim \mathcal{F}_0(\mathrm{ad}\ X) = \mathrm{rank}\ \mathcal{F}$ and $X$ is regular in $\mathcal{F}$.

__Lemma__ 1. If $X$ is regular, $\mathcal{G}_0(\mathrm{ad}\ X)$ is nilpotent.

__Proof__ $\mathcal{G}_0(\mathrm{ad}\ X)$ is a subalgebra in which $X$ is regular. Let $\mathcal{H} = \mathcal{G}_0(\mathrm{ad}\ X)$. Then $\mathcal{H}_0(\mathrm{ad}\ X) = \mathcal{H}$. Therefore, $\dim \mathcal{H}_0(\mathrm{ad}\ Y) \geq \dim \mathcal{H}$ for all $Y \in \mathcal{H}$. Therefore $\mathcal{H}_0(\mathrm{ad}\ Y) = \mathcal{H}$; i.e., $(\mathrm{ad}\ Y)^p_{\mathcal{H}} = 0$ for some $p$ -- whence $\mathcal{H}$ is nilpotent.

__Remark__ In fact, $\mathcal{G}_0(\mathrm{ad}\ X)$ is a Cartan subalgebra as

$$\mathcal{G}_0(\mathrm{ad}(\mathcal{G}_0(\mathrm{ad}\ X))) = \bigcap_{Y \in \mathcal{G}_0(\mathrm{ad}\ X)} \mathcal{G}_0(\mathrm{ad}\ Y) \subset \mathcal{G}_0(\mathrm{ad}\ X)$$

whence $\mathcal{G}_0(\mathrm{ad}\ \mathcal{H}) = \mathcal{H}$ where $\mathcal{H} = \mathcal{G}_0(\mathrm{ad}\ X)$.

Let $\mathcal{G}$ be a Lie algebra over an algebraically closed ground field $K$. Let $\mathcal{H}$ be a nilpotent Lie algebra in $\mathcal{E} = \mathrm{Hom}_k(V,V)$. (The elements of $\mathcal{H}$ need not be nilpotent endomorphisms of $V$). Suppose $\lambda: \mathcal{H} \longrightarrow K$

is a linear homomorphism of $\mathcal{H}$ to the additive group of K. Set

$$V_\lambda(\mathcal{H}) = \bigcap_{X \in \mathcal{H}} V_{\lambda(X)}\, X.$$

Call $\lambda$ a weight in $\mathcal{H}$ if and only if $V_\lambda \neq 0$. $V_\lambda$ is called the weight space of $\lambda$.

**Theorem** $V = \sum_\lambda V_\lambda$ (direct)

**Proof** $\mathcal{E}_a(\mathrm{ad}\ T)\ V_b(T) \subset V_{a+b}(T)$. For any $X, Y$ in $\mathcal{H}$, $y \in \mathcal{E}_0(\mathrm{ad}\ X)$. We have $Y\ V_b(X) \subset V_b(X)$. The procedure is now like the well-known simultaneous diagonalization of an abelian family of diagonalizable linear endomorphisms. That is, we decompose each $V_b(X)$ with respect to other elements of $\mathcal{H}$ etc. and ultimately arrive at subspaces $W_\alpha$ such that each element $X$ of $\mathcal{H}$ has only a single eigenvalue on $W_\alpha$. For each such $W$, let $\lambda_W(X)$ be the eigenvalue of $X$ on $W$ -- $\lambda_W(X)$ is linear in $X$. $W = V_\lambda(\mathcal{H})$ and $V = \Sigma\, V_\lambda(\mathcal{H})$.

We now begin the classification of semi-simple Lie algebras. We assume $K$ is of characteristic zero and algebraically closed (although much of the following holds more generally).

Let $\mathcal{H}$ be a nilpotent Lie algebra over the field $K$. Let $\lambda\colon \mathcal{H} \longrightarrow K$ be a homomorphism of $\mathcal{H}$ into the additive group of $K$. Let $\rho$ be a representation of $\mathcal{H}$ on the vector space $V$ over $K$; i.e., $\rho\colon \mathcal{H} \longrightarrow \mathrm{Hom}_K(V,V)$ is required to be a homomorphism of Lie algebras.

Definition $\rho V_\lambda(\mathcal{H}) = \bigcap_{X\in\mathcal{H}} V_{\lambda(X)}(\rho(X))$. When only one nilpotent algebra $\mathcal{H}$ and one representation $\rho$ is under discussion we usually write $V_\lambda$ for $\rho V_\lambda(\mathcal{H})$.

The linear functional $\lambda$ is called a weight of $\rho$ if $\rho V_\lambda(\mathcal{H}) \neq (0)$. (If $\mathcal{H}$ is a Cartan subalgebra of $\mathcal{G}$ and $\rho$ is the adjoint representation $\mathrm{ad}\colon \mathcal{H} \to \mathrm{Hom}_K(\mathcal{G},\mathcal{G})$ then the weights are called roots.)

The following theorem is a direct generalization of the theorem of Lecture XXI to the case of an arbitrary representation.

Theorem If $K$ is algebraically closed, then

$$V = \sum_\lambda V_\lambda \quad \text{(direct)}$$

The space $V_\lambda$ is stable under $\rho(X)$ for each $X \in \mathcal{H}$, and, if we set $\rho_{V_\lambda}(X) = V_\lambda$ — part of $\rho(X)$ (i.e., $\rho(X)$ restricted to $V_\lambda$) we have:

$$\mathrm{tr}\,\rho_{V_\lambda}(X) = \lambda(X)\cdot\dim V_\lambda .$$

Proposition For any $X, Y \in \mathfrak{H}$ and weight $\lambda$, we have

$$\lambda([X,Y]) = 0.$$

Proof Choose an eigenvector $v \in V_\lambda$ (this possibility is guaranteed by Lie's theorem on solvable Lie algebras over a field of characteristic 0 and algebraically closed).

Now:
$$\begin{aligned}\lambda([X,Y])v &= \rho[X,Y]v \\ &= [\rho(X),\rho(Y)]v \\ &= \rho(X)\rho(Y)v - \rho(Y)\rho(X)v \\ &= \lambda(X)\lambda(Y)v - \lambda(Y)\lambda(X)v \\ &= 0\end{aligned}$$

when $\lambda([X,Y]) = 0$.

In the classification of semi-simple algebras we will have need of a criterion for recognizing solvable ideals.

Basic Lemma. Let $\mathfrak{G}$ be a Lie algebra (over K) and $\rho$ a representation of $\mathfrak{G}$ on the K-space V. Assume

$$\operatorname{tr} \rho(X)\rho(Y) = 0$$

for all $X, Y$ in $\mathfrak{G}$.

Then $\rho(X)$ is nilpotent for each $X$ in $[\mathfrak{G},\mathfrak{G}]$.

Proof. Choose an element $X^*$ in $[\mathfrak{G},\mathfrak{G}]$ such that $\dim V_0(\rho(X^*))$ is minimal; i.e.,

$$\dim V_0(\rho(X^*)) \leq \dim V_0(\rho(Y)) \quad \text{for all } Y \in [\mathfrak{G},\mathfrak{G}]$$

and also require

$$\dim [\mathfrak{G},\mathfrak{G}]_0(\operatorname{ad} X^*) \leq \dim [\mathfrak{G},\mathfrak{G}]_0(\operatorname{ad} Y) \quad \text{for all } Y \in [\mathfrak{G},\mathfrak{G}].$$

(This is possible, as we need only pick $X^*$ on the

[illegible]

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complement of the union of two hypersurfaces.)

Set $\mathcal{H} = \mathcal{G}_0(\mathrm{ad}\, X^*)$. As $X^*$ is regular in the derived subalgebra $[\mathcal{G},\mathcal{G}]$ we know that $[\mathcal{G},\mathcal{G}] \cap \mathcal{H}$ is a Cartan subalgebra of $[\mathcal{G},\mathcal{G}]$. Set $\mathcal{G}_\alpha = \mathcal{G}_\alpha(\mathrm{ad}\, X^*)$ for $\alpha \in K$. Then

$$\mathcal{G} = \sum_{\alpha \in K} \mathcal{G}_\alpha .$$

We know, also:

1) $[\mathcal{G}_\alpha, \mathcal{G}_\beta] \subset \mathcal{G}_{\alpha+\beta}$
2) $\mathcal{G}_0 = \mathcal{H}$
3) $[\mathcal{G},\mathcal{G}] \cap \mathcal{H}$ is spanned by $[\mathcal{G}_\alpha, \mathcal{G}_{-\alpha}]$ and $[\mathcal{H},\mathcal{H}]$.

We set the following lemma off because of its importance for what follows now and certain considerations later.

Lemma. Let $X_\alpha \in \mathcal{G}_\alpha$, $X_{-\alpha} \in \mathcal{G}_{-\alpha}$ and set

$$h_\alpha = [X_\alpha, X_{-\alpha}].$$

For any weight $\lambda$ of the representation $\rho$

$$\lambda(h_\alpha) = \gamma_{\lambda,\alpha}\alpha(h_\alpha)$$

where $\gamma_{\lambda,\alpha}$ is a rational number.

Proof. Set $W = \sum_{n=-\infty}^{\infty} V_{\lambda+n\alpha}$. $W$ is stable under $\rho(X_\alpha)$ and $\rho(X_{-\alpha})$ $(\rho(X_\alpha): V_\mu \longrightarrow V_{\mu+\alpha})$.

The trace of $\rho(h_\alpha)_W$ is given by

$$\mathrm{tr}(\rho(h_\alpha)_W) = \mathrm{tr}[\rho(X_\alpha)_W, \rho(X_{-\alpha})_W] = 0.$$

On the other hand we have:

$$0 = \mathrm{tr}(\rho(h_\alpha)_W) = \sum_\alpha \mathrm{tr}(\rho(h_\alpha))_{V_{\lambda+n\alpha}}$$

$$= \dim V_{\lambda+n\alpha} \cdot [\lambda+n\alpha](h_\alpha)$$

from which the result follows immediately.

Now:

$$\begin{aligned} \operatorname{tr} \rho(h_\alpha)\rho(h_\alpha) &= \operatorname{tr} \rho(h_\alpha)^2 \\ &= \sum_\lambda \operatorname{tr} \rho(h_\alpha)^2_{V_\lambda} \\ &= \lambda(h_\alpha)^2 \cdot \dim V_\lambda \\ &= (\sum_\lambda \gamma_\lambda^2 \cdot \dim V_\lambda)\, \alpha(h_\alpha)^2 \\ &= 0. \end{aligned}$$

Since $(\sum_\lambda \gamma_\lambda^2 \dim V_\lambda)$ cannot vanish (characteristic $K = 0$) it follows that $\alpha(h_\alpha) = 0$ and thus:

$$\lambda(h_\alpha) = 0 \quad \text{for all} \quad \lambda.$$

Therefore, $\rho(h_\alpha)$ is nilpotent. From the proposition above we see that $\lambda(h) = 0$ for any weight $\lambda$ and element $h \in [\mathcal{H},\mathcal{H}]$ and thus $\rho(h)$ is nilpotent for $h \in [\mathcal{H},\mathcal{H}]$.

We see that $\mathcal{H} \cap [\mathcal{G},\mathcal{G}]$ is spanned by elements $\{X_i\}$ for which $\rho(X_i)$ is nilpotent. Thus $\rho(X)$ is nilpotent for all $X$ in $\mathcal{H} \cap [\mathcal{G},\mathcal{G}]$. But $X^* \in \mathcal{H} \cap [\mathcal{G},\mathcal{G}]$ and

$$\dim V = \dim V_0(\rho(X^*)) \leq \dim V_0(\rho(Y))$$

for all $Y \in [\mathcal{G},\mathcal{G}]$. Therefore $V_0(\rho(Y)) = V$ for $Y \in [\mathcal{G},\mathcal{G}]$ and $\rho(Y)$ is nilpotent for all $Y \in [\mathcal{G},\mathcal{G}]$. This concludes the proof of the Basic Lemma.

Corollary. If $\operatorname{tr} \operatorname{ad} X \operatorname{ad} Y = 0$ for all $X,Y \in [\mathcal{G},\mathcal{G}]$ then the Lie algebra $\mathcal{G}$ is solvable.

Proof. We apply the lemma to the Lie algebra $[\mathcal{G},\mathcal{G}]$ and the representation $\rho = \text{ad}\,\mathcal{G}$. Thus, $\rho(X)$ is nilpotent for all $X \in [[\mathcal{G},\mathcal{G}],[\mathcal{G},\mathcal{G}]] = \mathcal{G}_{(2)}$. Thus $\mathcal{G}_{(2)}$ is nilpotent. Thus $\mathcal{G}$ is solvable.

Definition.

Recall:

$$\det(\lambda - \text{ad}\,X) = P(\lambda, X) = \sum_k \lambda^{n-k} B_k(X)$$

where:

$B_2$ is a quadratic form on $\mathcal{G}$ $(B_2 = \sum_{i,j} \lambda_i\lambda_j$

$B_1$ is a linear form on $\mathcal{G}$ $(B_1 = \sum_i \lambda_i)$.

We use the name Killing form (a fundamental quadratic form) for $\text{tr}\,(\text{ad}\,X)^2 = \sum \lambda_i^2 = 1/2(B_2 - B_1^2)$. The polar form, $\text{tr ad}\,X\,\text{ad}\,Y$, is called the fundamental bilinear form.

Theorem. The Lie algebra $\mathcal{G}$ has no radical if and only if its fundamental bilinear form, $\text{tr ad}\,X\,\text{ad}\,Y$ is non-degenerate.

Proof Assume that $\text{tr ad}\,X\,\text{ad}\,Y$ is non-degenerate. Let $\mathfrak{a}$ be an abelian ideal in $\mathcal{G}$. For any $X$ in $\mathcal{G}$ and $Y$ in $\mathfrak{a}$ we have

$$\text{ad}\,X\,\text{ad}\,Y(\mathcal{G}) \subset \mathfrak{a},$$

whence $(\text{ad}\,X\,\text{ad}\,Y)^2 = 0$. Thus $\text{tr ad}\,X\,\text{ad}\,Y = 0$. It follows that $Y = 0$.

Let $\mathfrak{n} = \{Y;\ Y \in \mathcal{G}$ and $\text{tr ad}\,Y\,\text{ad}\,X = 0$ for all $X \in \mathcal{G}\}$. Set $B(X,Y) = \text{tr ad}\,X\,\text{ad}\,Y$. $B$ is invariant under the adjoint representation; i.e.,

for all $X,Y,Z \in \mathfrak{G}$ we have

$$B([Z,X],Y) + B[X,[Z,Y]] = 0$$

(this can be seen directly by looking at matrix representations of ad X, ad Y, ad Z).

This guarantees that $\mathfrak{N}$ is an ideal in $\mathfrak{G}$. Moreover, $B(X,Y) = 0$ for $X$ and $Y$ in $\mathfrak{N}$. By the Basic Lemma $[\mathfrak{N},\mathfrak{N}]$ is nilpotent, whence $\mathfrak{N}$ is solvable. Hence $\mathfrak{N}$ is (0) and tr ad X ad Y is non-degenerate.

Exercises

1. If $\mathfrak{G}$ is semi-simple, then $[\mathfrak{G},\mathfrak{G}] = \mathfrak{G}$.
2. If $\mathfrak{G}$ is semi-simple and D a derivation of $\mathfrak{G}$, then there is an $X \in \mathfrak{G}$ such that D = ad X.

Let $\mathcal{G}$ be a semi-simple Lie algebra over K. Consider ad: $\mathcal{G} \longrightarrow \mathrm{Hom}_K(\mathcal{G},\mathcal{G})$. Let $\mathcal{H}$ be a Cartan subalgebra. The weights for the adjoint representation of $\mathcal{H}$ on the underlying vector space of $\mathcal{G}$ are called roots. We have

$$\mathcal{G} = \Sigma\, \mathcal{G}_\alpha \quad \text{(direct)}$$

and the $\mathcal{G}_\alpha$ are called root spaces. We list, in this lecture, some of the fundamental properties of this decomposition.

Proposition 1. The restriction of $B = \operatorname{tr} \operatorname{ad} X \operatorname{ad} Y$ to $\mathcal{H} \times \mathcal{H}$ is non-degenerate.

Proof. $[\mathcal{G}_\alpha,\mathcal{G}_\beta] \subset \mathcal{G}_{\alpha+\beta}$. Hence, for $X_\alpha \in \mathcal{G}_\alpha$ and $X_\beta \in \mathcal{G}_\beta$ we have:

$$\operatorname{ad} X_\alpha \operatorname{ad} X_\beta : \mathcal{G}_\gamma \longrightarrow \mathcal{G}_{\gamma+\alpha+\beta} .$$

Then $(\operatorname{ad} X_\alpha \operatorname{ad} X_\beta)^n \mathcal{G}_\gamma \subset \mathcal{G}_{\gamma+n(\alpha+\beta)}$. In case $(\alpha+\beta) \neq 0$ we know that there is an integer n such that $\gamma+n(\alpha+\beta)$ is not a root; and, therefore, $(\operatorname{ad} X_\alpha \operatorname{ad} X_\beta)^n = 0$. Thus $\operatorname{tr} \operatorname{ad} X \operatorname{ad} Y = 0$ and $B(\mathcal{G}_\alpha,\mathcal{G}_\beta) = 0$ if $(\alpha+\beta) \neq 0$. Setting $\alpha = 0$ we get $B(\mathcal{G}_0 = \mathcal{H},\mathcal{G}_\beta) = 0$ provided $\beta \neq 0$. If an element $h_0 \in \mathcal{H}$ is perpendicular to every element of $\mathcal{H}$ (i.e., $B(h_0,H) = 0$) then $B(h_0,\mathcal{G}) = 0$ and $h_0 = 0$.

Proposition 2. The number of linearly independent roots is the dimension of $\mathcal{H}$.

Proof. For $h \in \mathcal{H}$ we know $\mathcal{G}_\alpha$ is invariant under ad h. Furthermore the formula for the trace of ad h

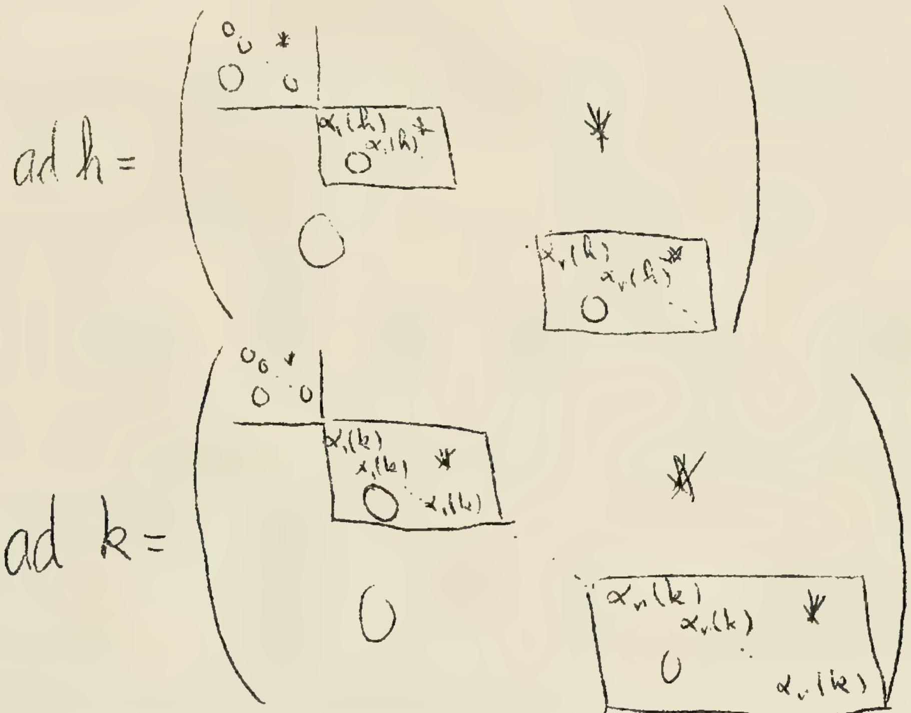
in $\mathcal{G}_\alpha$ is

$$\operatorname{tr}(\operatorname{ad} h)_{\mathcal{G}_\alpha} = (\dim \mathcal{G}_\alpha)\alpha(h).$$

For $h,k \in \mathcal{H}$ we have

$$B(h,k) = \operatorname{tr} \operatorname{ad} h \operatorname{ad} k = \sum_\alpha (\dim \mathcal{G}_\alpha)\alpha(h)\alpha(k)$$

because, since $\mathcal{H}$ is solvable, we may write (with respect to some basis for $\operatorname{ad} \mathcal{G}$).



Now, if the number of linearly independent roots is less than $\dim \mathcal{H}$ (as the roots are linear functionals on $\mathcal{H}$ the number of independent roots is less than or equal to the dimension of $\mathcal{H}^*$ which is equal to the dimension of $\mathcal{H}$) then there exists an element $h_0$ such that $\alpha(h_0) = 0$ for all roots $\alpha$. From the formula above we see $B(h_0, H) = 0$, and (by Proposition 1) $h_0 = 0$.

Proposition 3. The algebra $\mathcal{H}$ is abelian.

Proof. $\alpha([h,k]) = 0$ for all $h,k \in \mathcal{H}$.

Proposition 4. For any root $\alpha$ let $h_\alpha$ be defined by

$$B(h_\alpha, h) = \alpha(h) .$$

Then, for any $X_{-\alpha} \in \mathcal{G}_{-\alpha}$, $X_\alpha \in \mathcal{G}_\alpha$ with $[h, X_\alpha] = \alpha(h) X_\alpha$ we have $[X_\alpha, X_{-\alpha}] = B(X_\alpha, X_{-\alpha}) h_\alpha$.

Proof.

$$\begin{aligned} B([X_\alpha, X_{-\alpha}], h) &= -B(X_{-\alpha}, [X_\alpha, h]) \\ &= B(X_{-\alpha}, \alpha(h) X_\alpha) \\ &= \alpha(h) B(X_{-\alpha}, X_\alpha). \end{aligned}$$

Also, $B(B(X_\alpha, X_{-\alpha}) h_\alpha, h) = \alpha(h) B(X_\alpha, X_{-\alpha})$ and, since $B$ is non-degenerate we get

$$[X_\alpha, X_{-\alpha}] = B(X_\alpha, X_{-\alpha}) h_\alpha .$$

Remark. We prove below that $\dim \mathcal{G}_\alpha = 1$ if $\alpha \neq 0$. Then $X_\alpha$ can be replaced by any scalar multiple of $X_\alpha$ in the formula and we obtain

$$[X_\alpha, X_{-\alpha}] = B(X_\alpha, X_{-\alpha}) h_\alpha$$

for all $X_\alpha \in \mathcal{G}_\alpha$, $X_{-\alpha} \in \mathcal{G}_{-\alpha}$.

Proposition 5. If $\alpha$ is a root, then $-\alpha$ is also a root. If $k$ is an integer then $\dim \mathcal{G}_{k\alpha}$ is 0 if $|k| > 1$ and $\dim \mathcal{G}_\alpha = \dim \mathcal{G}_{-\alpha} = 1$.

Proof.

1) If $\alpha$ is a root, then $\mathcal{G}_\alpha \neq 0$. Hence $B(\mathcal{G}_\alpha, \mathcal{G}_\beta) = 0$ if $\beta \neq -\alpha$ implies $\mathcal{G}_{-\alpha} \neq 0$, lest $B(\mathcal{G}_\alpha, \mathcal{G}) = 0$.

2) Given the root $\alpha$, we take $X_\alpha \in \mathcal{G}_\alpha$ such that

$$[h, X_\alpha] = \alpha(h) X_\alpha \quad \text{for all } h \in \mathcal{H}.$$

Let $X_{-\alpha} \in \mathcal{G}_{-\alpha}$ be chosen so that $B(X_\alpha, X_{-\alpha}) \neq 0$. Then $h' = [X_\alpha, X_{-\alpha}]$ is in $\mathcal{H}$ and since $\beta(h')$

is a rational multiple of $\alpha(h')$ for all roots $\beta$, $\alpha(h')$ cannot vanish ($h'$ is, by construction, not equal to zero).

Now consider the space

$$V = KX_\alpha \oplus \mathcal{H} \oplus \sum_{k=1}^{\infty} \mathcal{G}_{-k\alpha} .$$

$V$ is invariant under ad $\mathcal{H}$, ad $X_\alpha$, ad $X_{-\alpha}$. We have

$$\operatorname{tr} (\operatorname{ad} h')_V = \operatorname{tr} [(\operatorname{ad} X_\alpha)_V, (\operatorname{ad} X_{-\alpha})_V] = 0$$

$$= \alpha(h')(1 + 0 - \sum_{k=1}^{\infty} k \dim \mathcal{G}_{-k\alpha} )$$

As $\alpha(h') \neq 0$ we must have $(1 - \sum_{k=1}^{\infty} k \dim \mathcal{G}_{-k\alpha}) = 0$ which implies $\dim \mathcal{G}_{-\alpha} = 1$, $\dim \mathcal{G}_{-k\alpha} = 0$ for $k > 1$. By reversing the role of $\alpha$ and $-\alpha$ we get the result requested.

Proposition 6. If $\mathcal{G}$ is semi-simple, then $[\mathcal{G},\mathcal{G}] = \mathcal{G}$.

Proof. Let $Z \in \mathcal{G}$. Then $B([\mathcal{G},\mathcal{G}],Z) = 0$ implies $B(X,[Z,Y]) = 0$ for all $X,Y \in \mathcal{G}$. This in turn implies $[Z,Y] = 0$ for all $Y$ and thus $Z = 0$. Thus the orthogonal complement of $[\mathcal{G},\mathcal{G}]$ is $(0)$ and $[\mathcal{G},\mathcal{G}] = \mathcal{G}$.

Proposition 7. Set

$$\begin{cases} q_{\beta,\alpha} = \text{largest positive integer } k \text{ such that } \beta + k\alpha \text{ is a root.} \\ \gamma_{\beta,\alpha} = \text{largest positive integer } k \text{ such that } \beta - k\alpha \text{ is a root.} \end{cases}$$

Then $\beta(h_\alpha) = 1/2(\gamma_{\beta,\alpha} - q_{\beta,\alpha})\, \alpha(h_\alpha)$ and $\beta + k_\alpha$ is a root for $-\gamma_{\beta,\alpha} \leq k \leq q_{\beta,\alpha}$.

Proof. Let $s$ and $t$ be a pair of integers with $s \leq t$ and such that $\beta + k_\alpha$ is a root for $s \leq k \leq t$ but

not for $k = s-1$ or $k = t+1$.

Consider the space $W$, stable under $\operatorname{ad} X_\alpha$ and $\operatorname{ad} X_{-\alpha}$, defined by:

$$W = \sum_{k=s}^{t} \mathfrak{g}_{\beta+k\alpha} :$$

We have:

$$0 = \operatorname{tr}[(\operatorname{ad} X_\alpha)_W, (\operatorname{ad} X_{-\alpha})_W] = \operatorname{tr}\,(\operatorname{ad}[X_\alpha, X_{-\alpha}])_W$$

$$= \operatorname{tr}\,(\operatorname{ad} h_\alpha)_W = \sum_{k=s}^{t} \dim \mathfrak{g}_{\beta+k\alpha}, \ [\beta+k\alpha](h_\alpha)$$

$$= \sum_{k=s}^{t} [\beta+k\alpha](h_\alpha) .$$

Thus,

$$(t-s+1)\beta(h_\alpha) = -[s+(s+1)+(s+2)+\dots+(s+t-s)]\alpha(h_\alpha)$$

$$= -[(t-s+1)s + \tfrac{1}{2}(t-s)(t-s+1)]\alpha(h_\alpha)$$

$$= -(t-s+1)[s+ \tfrac{1}{2}(t-s)]\alpha(h_\alpha),$$

whence,

$$\beta(h_\alpha) = -\tfrac{1}{2}(s+t)\alpha(h_\alpha) .$$

Given another such pair $(s', t')$ we cannot have:

a) $s' < s$ and $t < t'$, for then

$$(s+t') = s' + t',$$

by the above argument, and this is a contradiction to $s' < s$.

b) $s' < s$ and $t' < t$, for then

$$(s+t') = s+t,$$

and $t = t'$.

c) $s < s'$ and $t < t'$, for then

$$(s'+t) = s+t,$$

and $s' = s$.

d) $s < s'$ and $t' < t$, for then

$$(s+t') = s+t$$

and $t' = t$.

Thus, the only possubility is that $(s',t') = (s,t)$ $= (-r_{\beta,\alpha}, q_{\beta,\alpha})$ and the roots of the form $\beta + k_{\alpha}$ lie in a connected string.

The restriction of the fundamental bilinear form $B$ to $\mathcal{H}$ is non-degenerate and hence induces a bilinear form on the dual space $\mathcal{H}^*$. Explicitly, for $\lambda \in \mathcal{H}^*$, we define $h_\lambda \in \mathcal{H}$ by

$$B(h_\lambda, h) = \lambda(h) \quad \text{for all } h \in \mathcal{H}.$$

We set $\langle \lambda,\lambda' \rangle = B(h_\lambda, h_{\lambda'}) = \lambda(h_{\lambda'}) = \lambda'(h_\lambda)$ for $\lambda,\lambda'$ in $\mathcal{H}^*$.

Proposition 8. Let $\alpha,\beta$ be roots. Then $\langle \beta,\alpha \rangle$ is a rational number.

Proof. $\langle \alpha,\alpha \rangle = B(h_\alpha, h_\alpha) = \mathrm{tr}\,(\mathrm{ad}\, h_\alpha^2) = \sum\limits_\beta \beta(h_\alpha)^2$.

Thus,

$$\langle \alpha,\alpha \rangle = \sum_\beta \frac{(r_{\beta,\alpha} - q_{\beta,\alpha})^2}{4} \langle \alpha,\alpha \rangle^2$$

and we get

$$\frac{1}{\langle \alpha,\alpha \rangle} = \frac{1}{4} \sum_\beta (r_{\beta,\alpha} - q_{\beta,\alpha} \rangle^2,$$

which is a rational number ($\langle \alpha,\alpha \rangle$ is not zero for then $\alpha(h_\alpha) = 0$ and $\beta(h_\alpha) = \frac{1}{2}(r_{\beta,\alpha} - q_{\beta,\alpha})\alpha(h_\alpha)$ is zero for all $\beta$). As

$$\frac{2\langle \beta,\alpha \rangle}{\langle \alpha,\alpha \rangle} = r_{\beta,\alpha} - q_{\beta,\alpha}$$

is rational, the result follows.

Let $\mathcal{H}_0^*$ be the vector space over the field of rational numbers $Q$ whose elements are all linear combinations (with

rational coefficients) of the roots. By virtue of Proposition 8 we know that B induces a non-degenerate bilinear form on $\mathcal{H}_0^*$.

Proposition 9. $\dim_Q H_0^* = \dim_K H^*$

Proof. Take a basis $\{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$ for $\mathcal{H}^*$ with each $\alpha_i$ a root. An arbitrary root $\beta$ may be written $\sum_{i=1}^{\ell} c_i\alpha_i$ with $c_i\alpha_i$ with $c_i \in K$.

Now,

$$< \beta, \alpha_j > = \sum_{i=1}^{\ell} c_i < \alpha_i, \alpha_j >$$

and as $< \beta, \alpha_j >$ and $< \alpha_i, \alpha_j >$ are rational for $i = 1, 2, \ldots, \ell$ and any $j$ and $\det (< \alpha_i, \alpha_j >)$ is not zero we may use Cramer's rule to conclude that $c_i$ is rational for $i = 1, 2, \ldots, \ell$.

Proposition 10. Let $\alpha, \beta$ be roots. Then

$$[\mathcal{G}_\alpha, \mathcal{G}_\beta] = \mathcal{G}_{\alpha+\beta} .$$

Proof. If $\alpha+\beta$ is not a root we have

$$[\mathcal{G}_\alpha, \mathcal{G}_\beta] \subset \mathcal{G}_{\alpha+\beta} = (0),$$

whence $$[\mathcal{G}_\alpha, \mathcal{G}_\beta] = \mathcal{G}_{\alpha+\beta} .$$

If $\alpha+\beta$ is a root we consider the space $W = \sum_k \mathcal{L}_{\alpha+k\beta}$ and compute $\operatorname{tr} \operatorname{ad}[X_\alpha, X_{-\alpha}]$. It results that $[\mathcal{G}_\alpha, \mathcal{G}_\beta] \neq 0$.

Proposition 11. If $c \in K$ and $\alpha$ and $\beta = c\alpha$ are both roots then $c = \pm 1$.

Proof. $$\frac{< \beta, \alpha >}{< \alpha, \alpha >} = \frac{c < \alpha, \alpha >}{< \alpha, \alpha >}$$

whence, $c$ is a half integer.

Moreover, $\alpha = c^{-1}\beta$ and thus

$$\frac{< \alpha, \beta >}{< \beta, \beta >} = \frac{c^{-1} < \beta, \beta >}{< \beta, \beta >}$$

one of $c$ and $c^{-1}$ is $2$. The latter possibility has already been excluded (see Proposition 5).

Exercise: Prove $\sum_{\text{all } \alpha} < \alpha,\alpha > = \dim \mathcal{H} = \operatorname{rank} \mathcal{G}$.

Hint (consider the matrix $< \beta,\alpha >$, all roots $\alpha$ and $\beta$).

# Lecture XXIV

In this lecture we discuss the representations of the three-dimensional simple Lie algebra. This will prove useful in the classification of semi-simple algebras, and it is of great importance as an example for the general theory of representations.

Let $\mathcal{T}$ be the three-dimensional Lie algebra with generators $h$, $e_+$, $e_-$ and structure defined by:

$$[h,e_+] = ae_+$$

$$[h,e_-] = -ae_-$$

$$[e_+,e_-] = h,$$

for any element $a$ of $K$ with $a \neq 0$. $\mathcal{H} = Kh$ is obviously a Cartan subalgebra. A direct computation of $B$ shows that it is a non-degenerate form, and hence, $\mathcal{T}$ is semi-simple. We may then write $\mathcal{T}$ as a direct sum of simple ideals; but since a simple Lie algebra is of dimension at least equal to three, we see that there is but one simple term -- i.e., $\mathcal{T}$ is itself a simple Lie algebra. We will prove the following basic lemma concerning representations of $\mathcal{T}$.

Lemma. Let $\rho : \mathcal{T} \longrightarrow \mathrm{Hom}_K(V,V)$ be a representation of $\mathcal{T}$ on the finite dimensional vector space $V$ over $K$. Let $V = \sum_\lambda V_\lambda$ be the decomposition of $V$ into weight spaces with respect to $\mathcal{H}$. Then $\lambda(h)$ is a rational multiple of $a$. Let $\Lambda$ be the weight such that $a^{-1}\Lambda(h)$ is maximum among the numbers $a^{-1}\lambda(h)$. Set $\Lambda(h) = ma$. Assume $\rho$

is irreducible. Then $\dim V_\lambda = 1$ for all $\lambda$ and $\dim V = 2m+1$. For any $v_0 \in V_\Lambda$ we set $v_k = [\rho(e_-)]^k v_0$ and then

$$\rho(e_+)\rho(e_-)v_k = (k+1)(m-k/2)av_k$$
$$= [(k+1)m - 1/2k(k+1)]av_k.$$

Conversely, if $\dim V_\Lambda = 1$, $\rho(e_-)^t V_\Lambda \neq 0$, and $\rho(e_+)V_\Lambda = \rho(e_-)^{t+1}V_\Lambda = 0$, then, $\sum_{k=0}^{t} V_{\Lambda-k\alpha}$ is an irreducible representation module.

Proof.

A) We begin by showing that $\lambda(h)$ is a rational multiple of $a$. Set $\alpha(h) = a$.

Let $W_{\lambda,\alpha}$ be the subspace of $V$ defined by:

$$W_{\lambda,\alpha} = \sum_k V_{\lambda+k\alpha}.$$

The space $W_{\lambda,\alpha}$ is stable under $\rho(e_+)$ and $\rho(e_-)$. We have

$$\operatorname{tr} \rho(h) = \operatorname{tr} \rho[e_+,e_-] = \operatorname{tr}[\rho(e_+),\rho(e_-)] = 0.$$

On the other hand,

$$\operatorname{tr} \rho(h) = \Sigma\ (\dim V_{\lambda+k\alpha})[\lambda+k\alpha](h).$$

Thus,

$\lambda(h) = r\,\alpha(h)$ where $r$ is a rational number.

If we assume $\rho$ irreducible we have, here, $W_{\lambda,\alpha} = V$.

B) We now prove the formula -

$$\rho(e_+)\rho(e_-)v = (k+1)(m-k/2)av.$$

Let $v_0 \in V_\Lambda$ be different from zero. Define vectors $\{v_k\}$ for $k = 0,1,\ldots$ by

$$v_k = \rho(e_-)^k v_0.$$

For $k = 0$ we get:

$$\rho(e_+)\rho(e_-)v_0 = \{[\rho(e_+),\rho(e_-)] + \rho(e_-)\rho(e_+)\}v_0 = mav_0;$$

that is, $\rho(e_+)v_1 = mav_0$.

For $k = 1$ we have:

$$\rho(e_+)\rho(e_-)v_1 = \{\rho(h) + \rho(e_-)\rho(e_+)\}v_1 = \{(m-1)a + ma\}v_1;$$

then, $\rho(e_+)v_2 = \{(m-1)a + m(a)\}v_1$.

At the $k^{th}$ stage we get,

$$\begin{aligned}\rho(e_+)\rho(e_-)v_k &= \{\rho(h) + \rho(e_-)\rho(e_+)\}v_k \\ &= \{(m-k) + k(m - \tfrac{(k-1)}{2})\}av_k \\ &= (k+1)(m - k/2)av_k.\end{aligned}$$

C) Thus $\rho(e_+)\rho(e_-)$ has the eigenvalue $(k+1)(m-k/2)$ if $v_k$ is not zero. As $V$ is finite dimensional we conclude that, for some $k$, $v_k = 0$ but when $v_k \neq 0$ and $(k+1)(m-k/2) \neq 0$ we know $v_{k+1} \neq 0$. It follows that $(k+1)(m-k/2) = 0$ for some $k = 0,1,2,\ldots$. This means $m = k/2$ is a half-integer. Set

$$d = \sup\{k; v_k \neq 0\}$$

and let $W'$ be the subspace

$$W' = \sum_{k=0}^{d} Kv_k .$$

$W'$ is stable under $\rho(e_+)$ and $\rho(e_-)$ and hence under $\rho(h)$. Because $V$ is irreducible we have:

$$V = \sum_{k=0}^{d} Kv_k$$

and $d = 2m$. Thus each weight space is of dimension equal to 1 and $\dim V = 2m + 1$. This proves the lemma for the case that $\rho$ is irreducible. The

formula of B is valid under the hypothesis $\dim V_\Lambda = 1$, $\rho(e_+)V_\Lambda = \rho(e_-)^{2m+1}V_\Lambda = 0$, $\rho(e_-)^{2m}V_\Lambda \neq 0$. It follows from these formulae that V is irreducible.

We apply the formula above to the case of the adjoint representation to get the following:

Proposition. Let $\alpha$ be the root corresponding to the adjoint representation of $\mathcal{T}$; i.e., $\alpha(h) = a$. Let

$$\begin{cases} q_{\lambda,\alpha} = \sup\{n;\ \lambda+n\alpha \text{ is a weight}\} \\ r_{\lambda,\alpha} = \sup\{n;\ \lambda-n\alpha \text{ is a weight}\} \end{cases}$$

Then

1) $\rho(e_+)\rho(e_-)v_\lambda = 1/2(1+q_{\lambda,\alpha})r_{\lambda,\alpha}av_\lambda$

2) $\rho(e_-)\rho(e_+)v_\lambda = 1/2(1+r_{\lambda,\alpha})q_{\lambda,\alpha}av_\lambda$

where $v_\lambda \in V_\lambda$ .

Proof. We write $\lambda = \Lambda - k\alpha$ and then

$$\begin{cases} q_{\lambda,\alpha} = k \\ r_{\lambda,\alpha} = 2m - k \end{cases}$$

The result follows now directly from the lemma above.

Corollary. Let $\mathcal{G}$ be a semi-simple Lie algebra and

$$\mathcal{G} = \Sigma\, \mathcal{G}_\alpha$$

the decomposition of $\mathcal{G}$ into root spaces. Let $\alpha$ and $\beta$ be roots and let $\mathcal{T}$ be the 3-dimensional subalgebra $\mathcal{T} = \{X_\alpha, X_{-\alpha}, h_\alpha\}$. Let $\rho$ be the adjoint representation of $\mathcal{T}$ on $V = \sum_k \mathcal{G}_{\beta+k\alpha}$. Applying the result of the above proposition we have:

$$[X_{-\alpha},[X_\alpha,X_\beta]] = 1/2(1 + r_{\beta,\alpha})q_{\beta,\alpha} < \alpha,\alpha > X_\beta.$$

provided $\rho$ is irreducible. This will be the case except when $\alpha + \beta = 0$ and we have $V = \sum_k \mathcal{G}_{\beta+k\alpha}$ $\mathcal{G}_0 + \mathcal{G}_{-\alpha} + \mathcal{G}_\alpha$ having invariant subspaces $\mathcal{G}_{-\alpha}$ and $\mathcal{G}_\alpha$ for example.

Let $\{\rho_1, \rho_2, \ldots, \rho_\ell\}$ be a fixed basis of the vector space $\mathcal{H}_0^*$. With respect to this basis, we introduce a lexicographic ordering in $\mathcal{H}_0^*$; i.e., we say that $\sum_{i=1}^{\ell} c_i\rho_i = \rho \geq 0$ just in case the first non-vanishing coefficient $c_j$ is non-negative. Note that $\rho_1 < \rho_2$ and $\sigma_1 < \sigma_2$ implies $\rho_1 + \sigma_1 < \rho_2 + \sigma_2$.

Definition. A positive root is called simple if it cannot be written as the sum of two positive roots. We denote by $\Pi$ the set of simple positive roots.

Proposition 1

1) If $\alpha, \beta \in \Pi$ and $\alpha \neq \beta$ then $\alpha - \beta$ is not a root.

2) Under the same hypotheses we have $< \alpha, \beta > \leq 0$.

3) $\Pi$ is a basis for $\mathcal{H}_0^*$ (over Q) and each positive root $\alpha$ can be written as

$$\alpha = \sum_{\alpha_i \in \Pi} m_i\alpha_i$$

with $m_i$ a non-negative integer for each $i$. Moreover, $\alpha - \alpha_i$ is a root for some $\alpha_i \in \Pi$.

Proof.

1) If $\alpha - \beta$ were a root, we would have either $\alpha - \beta$ or $\beta - \alpha$ a positive root. Say $\gamma = \beta - \alpha$ is positive. Then $\beta = \gamma + \alpha$ and $\beta$ is written as the sum of two positive roots.

2) By virtue of 1) we have

$$\frac{2 < \beta, \alpha >}{< \alpha, \alpha >} = r_{\beta,\alpha} - q_{\beta,\alpha} = -q_{\beta,\alpha} \leq 0.$$

Since $< \alpha,\alpha > \geq 0$, it follows that

$$< \beta,\alpha > \leq 0.$$

3) We first prove the following:

**Lemma.** Let $S$ be a subset of $k$ positive elements in $\mathcal{H}_0^*$ such that

$$< \rho_i,\rho_j > \leq 0$$

for all $\rho_i,\rho_j$ in $S$ with $\rho_i \neq \rho_j$. Then the elements of $S$ form a linearly independent set.

**Proof.** In the contrary case, there is an element $\rho \in S$ which can be expressed

$$\rho = \sum_{i=1}^{S} \lambda_i\rho_i - \sum_{j=s+1}^{k} \lambda_j\rho_j \, .$$

where $\lambda_i,\lambda_j$ are $\geq 0$, the $\rho_i$ run over all elements of $S$, the coefficient of $\rho$ on the right-hand side is zero, and, say, $\lambda_1 > 0$.

Then

$$< \rho, \sum_{i=1}^{S} \lambda_i\rho_i > = \sum_{i=1}^{S} \lambda_i < \rho,\rho_i > \leq 0.$$

On the other hand,

$$\left\langle \sum_{i=1}^{S} \lambda_i\rho_i - \sum_{S+1}^{k} \lambda_j\rho_j, \sum_{i=1}^{S} \lambda_i\rho_i \right\rangle$$

$$= \left\langle \sum_{i=1}^{S} \lambda_i\rho_i, \sum_{i=1}^{S} \lambda_i\rho_i \right\rangle - \sum_{\substack{1\leq i\leq S \\ S+1\leq j\leq k}} \lambda_i\lambda_j < \rho_i,\rho_j >$$

and this is definitely positive, since it is a sum of non-negative terms the first of which is greater than zero ($B$ is positive definite on $\mathcal{H}_0^*$). This proves the lemma.

We can now prove 3). We assume the positive roots

arranged in order and proceed by induction.

Assume 3) has been proved for all positive roots less than the positive root $\beta$. Suppose it is the case that $\beta - \alpha$ is not a root for each $\alpha \in \Pi$; i.e., $r_{\beta,\alpha} = 0$ for all $\alpha \in \Pi$. Thus,

$$\frac{2<\beta,\alpha>}{<\alpha,\alpha>} = -q_{\beta,\alpha} \leq 0 \quad \text{for all} \quad \alpha \in \Pi$$

and the set $\{\beta\} \cup \Pi$ is seen to be a linearly independent set. Now $\beta$ is not simple (else there is nothing to prove) so we write $\beta = \beta_1 + \beta_2$ with $\beta_1$ and $\beta_2$ both positive roots. By induction $\beta = \beta_1 + \beta_2 = \sum_{\alpha_i \in \Pi} m_i\alpha_i$ with $m_i \geq 0$. Hence, if $\beta \notin \Pi$ then $\beta - \alpha$ is a root for some $\alpha \in \Pi$ -- $\beta - \alpha$ must be positive if it is a root, since $\alpha$ is simple.

Proposition 2. Let $\mathcal{S}$ be the subset of $\mathcal{G}$ consisting of $\{h_\alpha, e_\alpha, e_{-\alpha}; \alpha \in \Pi$ and $B(e_\alpha, e_{-\alpha}) = 1\}$. The smallest subalgebra of $\mathcal{G}$ which contains $\mathcal{S}$ is $\mathcal{G}$ itself.

Proof. Given any positive root $\beta$ there is a chain of simple roots $\alpha_1, \alpha_2, \ldots, \alpha_r$ (not necessarily distinct) such that

$$\alpha_1, \alpha_1+\alpha_2, \alpha_1+\alpha_2+\alpha_3, \ldots, \sum_{i=1}^{r} \alpha_i = \beta$$

are all roots. Then

$$[\ldots[[\mathcal{G}_{\alpha_1}, \mathcal{G}_{\alpha_2}], \mathcal{G}_{\alpha_3}]\ldots] = \mathcal{G}_\beta.$$

Thus each root space is in the span of $\mathcal{S}$ and $\mathcal{H}$ is even in the linear span of $\mathcal{S}$. This proves Proposition 2.

Let $\{\alpha_1,\alpha_2,\dots,\alpha_\ell\} = \Pi$. Consider the matrix $(A_{ij})$ where:

$$A_{ij} = \frac{2<\alpha_j,\alpha_i>}{<\alpha_i,\alpha_i>} = -q_{\alpha_j,\alpha_i} \leq 0 .$$

The matrix $(A_{ij})$ is called the matrix of Cartan integers of $\mathcal{G}$. In the following lectures it will be shown that $(A_{ij})$ determines the structure of $\mathcal{G}$ and we will classify those matrices which arise as matrices of Cartan integers.

# Lecture XXVI

We devote this chapter to the proof that $(A_{ij})$ determines the structure of $\mathfrak{g}$. We begin by showing how the Cartan integers determine the set of roots.

__Definition__. Let $\beta = \sum_{i=1}^{\ell} m_i \alpha_i$ be a positive root, i.e., $m_i$ is a non-negative integer for $i = 1,2,\ldots,\ell$. We define the order of $\beta$ by

$$\text{order of } \beta = O(\beta) = \sum_{i=1}^{\ell} m_i \, .$$

__Proposition 1__. Given a system of simple roots $\Pi = \{\alpha_1,\ldots,\alpha_\ell\}$ and the $\ell \times \ell$ matrix of Cartan integers $(A_{ij})$, the set of all roots is completely determined.

__Proof__. The elements of order $1$ are all roots -- in fact, simple roots. Suppose that we know all positive roots $\alpha$ such that $O(\alpha) \leq k$. Let $\beta$ be a positive element with $O(\beta) = k + 1$. Then, if $\beta$ is a root, there is an $\alpha_i$ such that $\gamma = \beta - \alpha_i$ is a root of order $k$. If, for each $i$, $\beta - \alpha_i$ is not a root then $\beta$ is not a root. If, on the other hand, we can write $\beta - \alpha_i = \gamma$ as a root of order $k$ the question of whether or not $\beta = \gamma + \alpha_i$ is a root is the question as to whether or not $q_{\gamma,\alpha_i}$ is non-zero. Writing

$$\frac{2\langle \gamma,\alpha_i \rangle}{\langle \alpha_i,\alpha_i \rangle} - r_{\gamma,\alpha_i} = -q_{\gamma,\alpha_i}$$

and observing that $r_{\gamma,\alpha_i}$ is determined, once we know all positive roots $\alpha$ of order $O(\alpha) \leq k - 1$, we see that $q_{\gamma,\alpha_i}$ is computable inductively.

Thus the Cartan integers $(A_{ij})$ for the system of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ tell us, for any roots $\alpha$ and $\beta$, the values of $r_{\beta,\alpha}$ and $q_{\beta,\alpha}$. By virtue of the formula:

$$\frac{4}{<\alpha,\alpha>} = \sum_\beta (r_{\beta,\alpha} - q_{\beta,\alpha})^2$$

we have $<\alpha,\alpha>$ determined. Then using

$$\frac{2<\beta,\alpha>}{<\alpha,\alpha>} = r_{\beta,\alpha} - q_{\beta,\alpha}$$

we read off $<\beta,\alpha>$. Thus the Cartan matrix $(A_{ij})$ determines the Killing form on $\mathcal{H}^*$ and, hence, on $\mathcal{H}$.

We now prove the fundamental

Theorem. The Cartan matrix $(A_{ij})$ (together with $\Pi$) determine the structure of $\mathcal{G}$.

Proof. Set $e_i = e_{\alpha_i}$ and $f_i = e_{-\alpha_i}$. For each positive root $\beta$ choose a sequence $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}$ of elements of $\Pi$ such that $\beta_S = \sum_{\ell=1}^{S} \alpha_{i_\ell}$ is a root for $S = 1, 2, \ldots, k$.

For convenience we write $[X_k X_{k-1} \ldots X_1]$ for $[X_k[X_{k-1}[\ldots[X_2, X_1]\ldots]$. Then set

$$e_\beta = [e_{i_k} \ldots e_{i_1}]$$

$$e_{-\beta} = [f_{i_k} \ldots f_{i_1}] \; .$$

The elements $\{h_{\alpha_i}; e_\beta, e_{-\beta}\}$ form a basis for the underlying vector space of $\mathcal{G}$. To show that $\mathcal{G}$ is determined by $(A_{ij})$ we need to show that the structure constants with respect to this basis are so determined.

Proposition 2. Let $e_\beta = [e_{i_k} \dots e_{i_1}]$ and let $\sigma$ be a (fixed) permutation of the indices $\{1,\dots,k\}$. Then

$$[e_{i_{\sigma(k)}} \dots e_{i_{\sigma(1)}}] = c_\sigma e_\beta$$

where $c_\sigma$ is a rational number and is determined by $\sigma$ and the Cartan matrix $(A_{ij})$.

Proof. We argue by induction on $k$. Having conceded the case $k=1$, we assume the Proposition proved for $k \leq \ell - 1$ and let $\beta = [e_{i_\ell} \dots \ell_{i_1}]$.

Case 1 Assume $\sigma(\ell) = \ell$. Then write

$$\beta = \alpha_\ell + \sum_{S=1}^{\ell-1} \alpha_{i_S} = \alpha_\ell + \beta'.$$

We may assume that

$$e_{\beta'} = [e_{i_{\ell-1}} e_{i_{\ell-2}} \dots e_{i_1}] .$$

Then the result for $k \leq \ell - 1$ gives

$$[e_{i_{\sigma(\ell-1)}} e_{i_{\sigma(\ell-2)}} \dots e_{i_1}] = c_1 e_{\beta'}$$

and $c_1$ is a rational number which depends on $(A_{ij})$ and the permutation $\sigma$ (actually on the restriction of $\sigma$ to $\{1,2,\dots,(\ell-1)\}$). We then have:

$$\begin{aligned}[e_{\sigma(\ell)} \dots e_{\sigma(1)}] &= [e_\ell,[e_{\sigma(\ell-1)} \dots e_{\sigma(2)}]] \\ &= c_1[e_\ell,[e_{\ell-1} \dots e_1]] \\ &= c_1 e_\beta .\end{aligned}$$

Case 2. We assume $\ell \neq \sigma(\ell)$. Then write

$$[e_{i_{\sigma(\ell)}} \dots e_{i_{\sigma(1)}}] = [e_{i_{\sigma(\ell)}} \dots e_{i_\ell} \dots e_{i_{\sigma(1)}}].$$

It may happen that not all partial sums $\sum_{S=1}^{t} \alpha_{i_{\sigma(S)}}$, $t \leq \ell$, are roots. In this case $[e_{i_{\sigma(\ell)}} \dots e_{i_{\sigma(1)}}]$

= 0 and we set $c_{\sigma} = 0$ -- as this phenomenon is determined by $(A_{ij})$, we are done in this case.

We assume, therefore, that all partial sums $\sum_{S=1}^{t} \alpha_{i_{\sigma(S)}}$, $t \leq \ell$, are roots; i.e., $[e_{i_{\sigma(\ell)}} \cdots e_{i_{\sigma(1)}}] \neq 0$.

Now, we may write

$$[f_{i_\ell},[e_{i_{\sigma(\ell)}} \cdots e_{i_{\sigma(1)}}]] = [e_{i_{\sigma(\ell)}} \cdots f_{i_\ell} e_{i_\ell} e_y \cdots e_{i_{\sigma(1)}}]$$

[since $[f_{i_\ell}, e_{i_k}] = 0$ when $k \neq \ell$ ($\alpha_{i_k} - \alpha_{i_\ell}$ is not a root) we have ad $f_{i_\ell}$ ad $e_{i_k}$ = ad $e_{i_k}$ ad $f_{i_\ell}$ + ad $[f_{i_\ell}, e_{i_k}]$ = ad $e_{i_k}$ ad $f_{i_\ell}$]. Appealing to the Corollary of Lecture XXIV we have

$$[e_{i_\ell} f_{i_\ell} e_{i_{\sigma(\ell)}} \cdots e_{i_\ell} \cdots e_{i_{\sigma(1)}}] = c_2 [e_{i_{\sigma(\ell)}} \cdots e_{i_\ell} \cdots e_{i_{\sigma(1)}}]$$

with $c_2$ a rational number determined by $\sigma$ and $(A_{ij})$.

Similarly

$$[e_{i_\ell} f_{i_\ell} e_y \cdots e_{\sigma(1)}] = c_3 [e_y \cdots e_{\sigma(1)}]$$

and neither $c_2$ nor $c_3$ is zero.

Then

$$\begin{aligned} & c_2 [e_{i_{\sigma(\ell)}} \cdots e_{i_\ell} \cdots e_{i_{\sigma(1)}}] \\ & = [e_{i_\ell} f_{i_\ell} e_{i_{\sigma(\ell)}} \cdots e_{i_\ell} \cdots e_{i_{\sigma(1)}}] \\ & = c_3 [e_{i_\ell} e_{\sigma(\ell)} \cdots e_{i_{(y+2)}} e_{i_{(y)}} \cdots e_{\sigma(1)}] \end{aligned}$$

Thus

$$[e_{i_{\sigma(\ell)}} \cdots e_{i_{\sigma(1)}}] = c_3/c_2 [e_{i_\ell} e_{i_{\sigma(\ell)}} \cdots e_{i_{(y+2)}}, e_{i(y)} \cdots e_{i_{\sigma(1)}}]$$

and the right side can (by reduction to the case $\sigma(\ell) = \ell$) be written $c_4 e_\beta$.

**Theorem.** The constants of structure of $\mathcal{G}$ with respect to the basis $\{h_{\alpha_i}, e_\beta, e_{-\beta}\}$ are rational numbers determined by $(A_{ij})$.

**Proof.** Since $[h_i, h_j] = 0$ and since all inner products $< \alpha, \beta >$ are rational we have the result for products of the following types:

$$[h_i, h_j]$$
$$[h_j, [e_\ell \ldots e_1]]$$
$$[h_j, [f_\ell \ldots f_1]]$$

so that we need only prove that for $\gamma_1, \gamma_2$, arbitrary roots, we have $[e_{\gamma_1}, e_{\gamma_2}]$ a rational combination of the basis vectors. This can be reduced to proving the conclusion for the cases $[e_j, [e_{i_\ell}, \ldots, e_{i_1}]]$, $[e_j, [f_{i_\ell}, \ldots, f_{i_1}]]$, $[f_j [e_{i_\ell} \ldots e_{i_1}]]$, $[f_j, [f_{i_\ell} \ldots f_{i_1}]]$. We give the proofs for the first two cases, the others being analogous.

A) We show $[e_j, [e_{i_\ell} \ldots e_{i_1}]]$ is a rational combination of basis elements.

Proof Set $\beta = \alpha_{i_1} + \ldots + \alpha_{i_\ell}$. If $\beta + \alpha_j$ is not a root (which fact can be determined by $(A_{ij})$) we have $[e_j [e_{i_\ell} \ldots e_{i_1}]] = 0$.

If $\beta + \alpha_j$ is a root then $[e_j, [e_{i_\ell} \ldots e_{i_1}]]$ is a rational multiple (depending on $(A_{ij})$) of the base element $e_\gamma$ (by Proposition 2).

B) The proof for $T = [e_j,[f_{i_\ell},\dots,f_{i_1}]]$ runs as follows. For $\ell = 1$ we have

$$T = \begin{cases} 0 & i_1 \neq j \\ -h_j & i_1 = j \end{cases} .$$

For $\ell = 2$ we have $T = 0$ unless $j = i_1$ or $j = i_2$. Then

$$T = \begin{cases} [e_j[f_j,f_{i_1}]] & i_2 = j \\ [e_j,[f_{i_2},f_j]] & i_1 = j \end{cases}$$

For the first case we get:

$$[e_j,[f_j,f_{i_1}]] = 1/2\, q_{\alpha_i,\alpha_j} < \alpha_j,\alpha_j > f_{i_1}$$

by virtue of the Corollary of Lecture XXIV. This coefficient is determined by $(A_{ij})$, as we have seen.

For the second case:

$$\begin{aligned} [e_j,[f_{i_2} f_j]] &= -[f_j,[e_j,f_{i_2}]] - [f_{i_2},[f_j,e_j]] \\ &= 1/2\, q_{\alpha_{i_2},\alpha_j} < \alpha_j,\alpha_j > f_{i_2} - \alpha_{i_2}(h_j) f_{i_2} \\ &= -2 < \alpha_{i_2},\alpha_j > f_{i_2}. \end{aligned}$$

We proceed by induction. Let $\ell > 2$. If no $i_S = j$ the product is 0. In the contrary case let $i_{S+1}$ be the highest index in $[f_{i_\ell}\dots f_{i_1}]$ for which $i_{S+1} = j$. Then

$$\begin{aligned} &[e_j,[f_{i_\ell}\dots f_j f_{i_S}\dots f_{i_1}]] \\ &= [f_{i_\ell}\dots f_{i_{S+2}} [e_j,[f_j f_{i_S}\dots f_{i_1}]]] \\ &= [f_{i_\ell}\dots f_{i_{S+2}} [h_j,[f_{i_S}\dots f_{i_1}]]] \\ &= [[f_{i_\ell}\dots f_{iS+2} f_j e_j] f_{iS}\dots f_{i1}] \end{aligned}$$

and we apply the induction hypothesis to complete the proof.

Corollary 1. Let $\mathcal{G}$ and $\mathcal{G}'$ be semi-simple algebras and let $\mathcal{H}$ and $\mathcal{H}'$ be Cartan subalgebras, respectively. Let $\Pi$ and $\Pi'$ be systems of simple roots on $\mathcal{H}$ and $\mathcal{H}'$, respectively. Let $\varphi$: $\Pi \longrightarrow \Pi'$ be a bijection and assume

$$\frac{2 < \beta,\alpha >}{< \alpha,\alpha >} = \frac{2 < \varphi(\beta),\varphi(\alpha) >}{< \varphi(\alpha),\varphi(\alpha) >}$$

for all $\alpha,\beta$ in $\Pi$.

Then there is exactly one isomorphism $\varphi^*: \mathcal{G} \longrightarrow \mathcal{G}$ such that for $\alpha \in \Pi$

$$\varphi^*(h_\alpha) = h_{\varphi(\alpha)}$$
$$\varphi^*(e_\alpha) = e_{\varphi(\alpha)}$$
$$\varphi^*(e_{-\alpha}) = e_{-\varphi(\alpha)}$$

where $e_\alpha \in \mathcal{G}_\alpha$, $e_{-\alpha} \in \mathcal{G}_{-\alpha}$ with $B(e_\alpha, e_{-\alpha}) = 1 =$ tr (ad $e_\alpha$ ad $e_{-\alpha}$) = tr (ad $e_{\varphi(\alpha)}$, ad $e_{-\varphi(\alpha)}$).

Corollary 2. Let $\mathcal{G}$ be a semi-simple algebra. There is an automorphism of order 2 $\varphi^*$ of $\mathcal{G}$ such that $\varphi(h) = -h$ for all $h \in \mathcal{H}$, where $\mathcal{H}$ is a Cartan subalgebra.

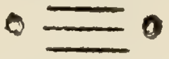
Proof. If $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ is a system of simple roots we set $\Pi' = \{-\alpha_1, -\alpha_2, \ldots, -\alpha_\ell\}$. The mapping $\varphi: \Pi \longrightarrow \Pi'$ defined by $\varphi(\alpha_i) = -\alpha_i$ enjoys the property.

$$\frac{< \alpha_i,\alpha_j >}{< \alpha_j,\alpha_j >} = \frac{< -\alpha_i,-\alpha_j >}{< -\alpha_j,-\alpha_j >} = \frac{< \varphi(\alpha_i),\varphi(\alpha_j) >}{< \varphi(\alpha_j),\varphi(\alpha_j) >}$$

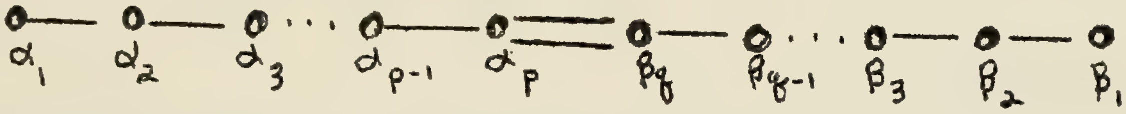
whence, by Corollary 1, we obtain an automorphism $\varphi^*$ of $\mathcal{G}$ such that $\varphi(h) = -h$ for all $h \in \mathcal{H}$.

# LECTURE XXVIII

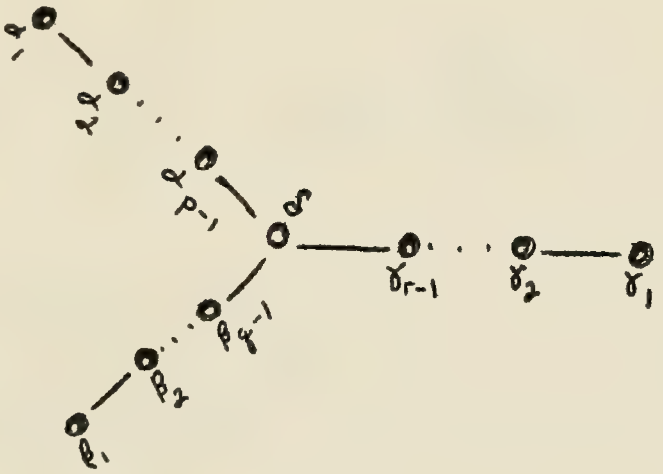
We have established that if the configuration of a Dynkin diagram (i. e., the diagram without the weights attached) has a triple joint it is the diagram



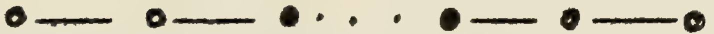
which we call $\underline{\underline{G_2}}$. If it has a double joint, it has but one double joint and is of the form

(i) 

and if it has no multiple joints it is either of the form

(ii) 

or else



which we denote $A_n$ (where $n$ is the number of elements in the system $\Pi$).

We discuss these types in more detail now.

## TYPE (i)

Limitations on $p$ and $q$.

Set $\alpha = \sum_{i=1}^{p} i\alpha_i$ and $\beta = \sum_{j=1}^{q} j\beta_j$. By Schwartz's inequality, we have

$$(*) \quad \langle\alpha,\beta\rangle^2 < |\alpha|^2|\beta|^2 ,$$

the strict inequality holding since $\alpha$ and $\beta$ are not collinear.

Now

$$|\alpha|^2 = \sum_{i=1}^{p} i^2|\alpha_i|^2 + 2\sum_{i=1}^{p-1} i(i+1)\langle\alpha_i,\alpha_{i+1}\rangle$$

$$= |\alpha_1|^2 \sum_{i=1}^{p} i^2 + \sum_{i=1}^{p-1} i(i+1)(-1)$$

$$= |\alpha_1|^2(\sum_{i=1}^{p-1} - i + p^2) = |\alpha_1|^2(p^2 - \frac{(p-1)p}{2})$$

$$= |\alpha_1|^2(\frac{p(p+1)}{2}).$$

Similarly

$$|\beta|^2 = |\beta_1|^2(\frac{q(q+1)}{2}).$$

Also

$$\langle\alpha,\beta\rangle = \langle p\alpha_p, q\beta_q\rangle = pq\langle\alpha_p,\beta_q\rangle .$$

The formula (*) above becomes, on substitution

$$p^2q^2\langle\alpha_p,\beta_q\rangle^2 < |\alpha_1|^2|\beta_1|^2 \frac{p(p+1)}{2}\frac{q(q+1)}{2} ,$$

whence

$$pq \frac{4\langle\alpha_p,\beta_q\rangle^2}{|\alpha_p|^2|\beta_q|^2} < (p+1)(q+1) .$$

Continuing, we get

$$2pq < (p+1)(q+1) = pq + p + q + 1$$

$$pq - p - q + 1 < 1 + 1$$

$$(p-1)(q-1) < 2$$

yielding the two cases:

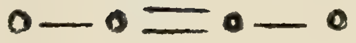
A) $(p-1)(q-1) = 1$

in which $p = 2 = q$

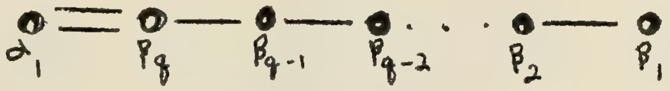
and B) $(p-1)(q-1) = 0$

in which $p = 1$ and $q$ is an arbitrary positive integer (the case $q = 1$ and $p$ arbitrary is isomorphic to this case).

The configuration is thus

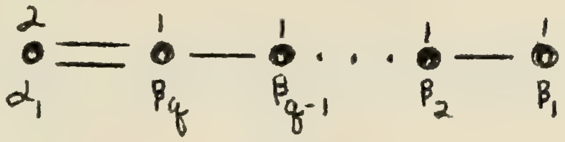


in case A). We call this configuration $F_4$. In case B) we have

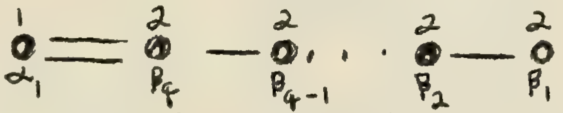


Allocation of weights in type (i)

Since the elements of a simple chain all have the same weights there are two types of Dynkin diagram corresponding to case B above; viz.,



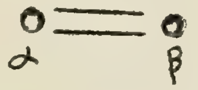
and



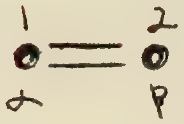
.

We call the first type $C_n$ and the second type $B_n$ where $n$ is the number of elements in $\Pi$.

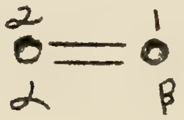
Note: In a diagram, if we have



we must have either



or



since

$$A_{\alpha\beta}A_{\beta\alpha} = 2 = \frac{4<\alpha,\beta>^2}{<\alpha,\alpha><\beta,\beta>}$$

and we have

$$A_{\alpha\beta} = -1 \quad A_{\beta\alpha} = -2$$

or else

$$A_{\beta\alpha} = -2 \quad A_{\alpha\beta} = -2$$

In the first case above we set $|\alpha|^2 = 2$ and get

$$\frac{2<\alpha,\beta>}{2} = -1$$

$$<\alpha,\beta> = -1$$

$$A_{\beta\alpha} = \frac{-2}{<\beta,\beta>} = -2$$

therefore $<\beta,\beta> = 1$

In the second case we set $|\alpha|^2 = 1$

$$\frac{2<\alpha,\beta>}{1} = -2$$

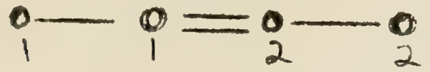
$$<\alpha,\beta> = -1$$

$$A_{\beta\alpha} = \frac{-2}{<\beta,\beta>} = -1$$

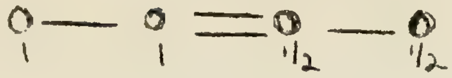
$$<\beta,\beta> = 2$$

From the note above we see that the allocation of weights in

the configuration $F_4$ must be of type



or



and these two diagrams are actually isomorphic (we may always change the weights by a constant multiple).

TYPE (ii) Limitations of p,q, and r

We assume that $p \geq q \geq r \geq 2$ (if $\min(p,q,r) = 1$ then we have the case $A_n$). Set

$$\alpha = \sum_{i=1}^{p-1} i\alpha_i$$

$$\beta = \sum_{j=1}^{q-1} j\beta_j$$

$$\gamma = \sum_{k=1}^{r-1} k\gamma_k \quad .$$

The vectors $\alpha, \beta, \gamma$ form a mutually orthogonal set (i. e., a 3-frame) in euclidean space. Denoting the angle between the vectors x and y by $\sphericalangle x,y$ we have:

$$\cos \sphericalangle \delta,\alpha = \frac{<\delta,\alpha>}{|\delta|\,|\alpha|} = \frac{(p-1)<\delta,\alpha_{p-1}>}{|\delta|\,|\alpha|} \quad .$$

On substituting

$$<\alpha,\alpha> = <\sum_{i=1}^{p-1} i\alpha_i, \sum_{i=1}^{p-1} i\alpha_i> = \sum_{i=1}^{p-1} i^2<\alpha_i,\alpha_i> + 2\sum_{i<j} ij<\alpha_i,\alpha_j>$$

$$= <\alpha_1,\alpha_1> \sum_{i=1}^{p-1} i^2 + 2\sum_{i=1}^{p-2} <\alpha_i,\alpha_{i+1}>i(i+1)$$

(cont. on next pg.)

$$= \langle\alpha_1,\alpha_1\rangle\Big(\sum_{i=1}^{p-1} i^2 - \sum_{i=1}^{p-2} i(i+1)\Big)$$

$$= \langle\alpha_1,\alpha_1\rangle\Big(\sum_{i=1}^{p-2} - i + (p-1)^2\Big)$$

$$= \langle\alpha_1,\alpha_1\rangle\Big(\frac{-(p-2)(p-1)}{2} + (p-1)^2\Big)$$

$$= \langle\alpha_1,\alpha_1\rangle(p-1)\Big((p-1) - \frac{(p-2)}{2}\Big)$$

$$= \langle\alpha_1,\alpha_1\rangle\frac{p(p-1)}{2}$$

we get

$$\cos^2 \sphericalangle\delta,\alpha = \frac{(p-1)^2\langle\delta,\alpha_{p-1}\rangle^2 2}{\langle\delta,\delta\rangle\langle\alpha_1,\alpha_1\rangle p(p-1)}$$

$$4\cos^2 \sphericalangle\delta,\alpha = \frac{2(p-1)^2}{p(p-1)} \cdot 4\frac{\langle\delta,\alpha_{p-1}\rangle}{\langle\delta,\delta\rangle\langle\alpha_1,\alpha_1\rangle} = \frac{2(p-1)}{p} = 2(1-\frac{1}{p})$$

The inequality

$$4\cos^2 \sphericalangle\delta,\alpha + 4\cos^2 \sphericalangle\delta,\beta + 4\cos^2 \sphericalangle\delta,\gamma < 4$$

becomes on substitution

$$2(1-\frac{1}{p}) + 2(1-\frac{1}{q}) + 2(1-\frac{1}{r}) < 4$$

whence

$$1 < \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 3r^{-1}$$

and we find $r = 2$. Then we have

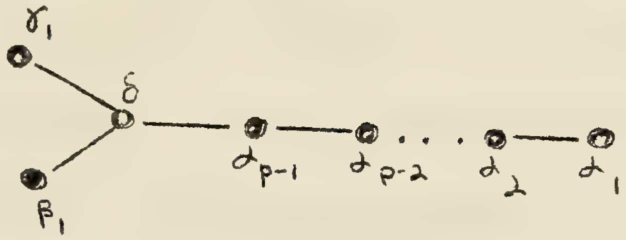
$$\frac{1}{2} < \frac{1}{p} + \frac{1}{q} \leq 2q^{-1}$$

whence $q = 2$ or $3$.

In case $q = 2$ we have

$$1 < \frac{1}{p} + \frac{1}{2} + \frac{1}{2}$$

and $p$ is arbitrary. The diagram, in this case is called $D_n$ and looks like

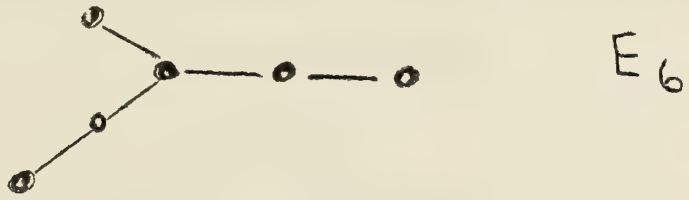


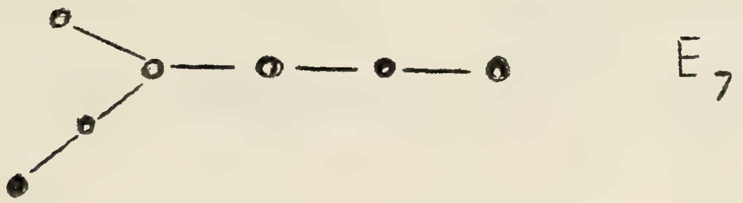
In case $q = 3$ we have

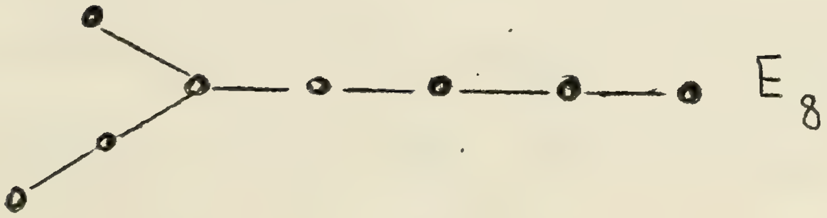
$$1 < \frac{1}{p} + \frac{1}{2} + \frac{1}{3}$$

$$3 \leq p < 6$$

and $p = 3, 4,$ or $5$. The diagrams are shown below and are called $E_6, E_7,$ and $E_8$ respectively:







### Allocation of Weights on Type (ii)

In $A_n$, $D_n$, $E_6$, $E_7$, $E_8$ all weights are the same in any given diagram since all weights in a simple chain are equal.

We have ascertained that the Dynkin diagram of a simple Lie algebra is of the form $A_n$, $B_n$, $C_n$, $D_n$, $E_6$, $E_7$, $E_8$, $G_2$, or $F_4$. We have not shown that all of these are actually possible, but only that no others are. We now remedy this situation. (For a more detailed account see Lie Algebras, by N. Jacobson, p. 135 and the following pages).

### Existence of the simple Lie algebras of type $A_n$

Consider the Lie group $SL(n+1,K)$, the group of all $(n+1)$ by $(n+1)$ matrices of determinant $= 1 (n \geq 1)$. The Lie algebra $\mathfrak{sl}(n+1,K)$ consists of all $(n+1)$ by $(n+1)$ matrices of trace $= 0$.

Let $e_{ij}$ be the matrix $(\delta_{\alpha i} \cdot \delta_{\beta j})$ whose component $\gamma_{\alpha,\beta}$ is $0$ unless $\alpha = i$, $\beta = j$ in which case it is 1. We have

$$e_{ij} e_{k\ell} = \delta_{jk} e_{i\ell} \cdot$$

The diagonal elements of $\mathfrak{sl}(n+1,K)$ form a Cartan subalgebra $\mathcal{H}$. Denote by $\lambda_i$ the linear functional $\lambda_i(\sum_{j=1}^{n+1} a_j e_{jj}) = a_i$. The formula

$$[h, e_{ij}] = (a_i - a_j) e_{ij}$$

where $h = \sum_{k=1}^{n+1} a_k e_{kk}$, shows that $e_{ij}$ is a root vector for the root $\lambda_i - \lambda_j$ (where $i \neq j$). The $h^2 + n$ linear functionals

$(\lambda_i - \lambda_j)$ for $i \neq j$ are the roots and we get a simple system of roots

$$\alpha_1 = \lambda_1 - \lambda_2$$

$$\alpha_2 = \lambda_2 - \lambda_3$$

$$\vdots$$

$$\alpha_n = \lambda_n - \lambda_{n+1},$$

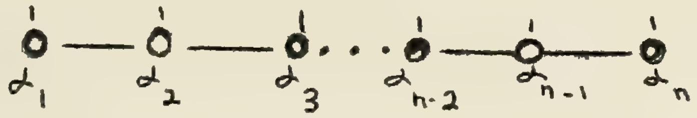
since, if $i < j$, we have $\lambda_i - \lambda_j = \alpha_i + \alpha_{i+1} + \dots + \alpha_{j-1}$. We have, then, the linear functionals $(\lambda_i - \lambda_j)$ with $i < j$ positive roots.

Thus

$$A\alpha_j\alpha_i = -q_{\alpha_j\alpha_i} = \begin{cases} 0 & \text{if } j \neq i+1 \text{ and } j \geq i \\ -1 & \text{if } j = i+1 \end{cases}$$

$$A_{\alpha_j\alpha_i} = A_{\alpha_i\alpha_j} \qquad \text{for } i > j$$

and the diagram of $\mathcal{S}l\,(n+1,K)$ is $A_n$:



We have shown that $A_n$ is the Dynkin diagram of a simple Lie algebra; viz., $\mathcal{S}l\,(n+1,K)$.

## Existence of the simple Lie algebras of type $B_n$

Consider the Lie group $SO(2n+1,K)$, the group of transformations of determinant = 1 which leave invariant the non-degenerate quadratic for F. Since, over an algebraically closed field, all (cont. on next page)

non-degenerate quadratic forms are equivalent, there is no loss of generality if we take the group to be all transformations preserving the quadratic form

$$x_0^2 + 2\sum_{i=1}^{n} x_i x_{i+n}$$

whose associated bilinear form is

$$x_0 y_0 + \sum_{i=1}^{n} x_i y_{i+n} .$$

This group is the set of all matrices of determinant = 1 whose elements $g$ satisfy

$$g\sigma\,{}^t g = \sigma$$

where

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0_n & 1_n \\ 0 & 1_n & 0_n \end{pmatrix}$$

is the matrix of the bilinear form.

The Lie algebra $\mathcal{SO}(2n+1,K)$ consists of all matrices $X$ such that

$$\exp(tX)\sigma\,{}^t\exp(tX) = \sigma;$$

i. e.,

$$(I+uX)\sigma^{t}(I+uX) = \sigma \quad \text{for all } u, \text{ i.e.,}$$

$$X\sigma + \sigma^{t}X = 0$$

Then $\mathcal{SO}(2n+1,K)$ consists of matrices

$$\begin{pmatrix} 0 & v_1 & v_2 \\ -{}^t v_2 & a_{11} & a_{12} \\ -{}^t v_1 & a_{21} & -{}^t a_{11} \end{pmatrix}$$

where $a_{12} = -{}^t a_{12}$, $a_{21} = -{}^t a_{21}$.

Take $\mathcal{H}$ = {diagonal elements in $\mathcal{SO}(2n+1,K)$} for a Cartan subalgebra. $\mathcal{H}$ is spanned by the elements $e_{i+1,i+1} - e_{i+n+1,i+n+1}$. We have the root vectors:

$$e_{\lambda i-\lambda j} = e_{j+1,i+1} - e_{i+n+1,j+n+1} \quad \text{for } i \neq j$$

$$e_{\lambda i+\lambda j} = e_{i+n+1,j+1} - e_{i+n+1,i+1} \quad \text{for } i < j$$

$$e_{-(\lambda_i+\lambda_j)} = e_{j+1,i+n+1} - e_{i+1,j+n+1} \quad \text{for } i < j$$

$$e_{\lambda_i} = e_{i,i+1} - e_{i+n+1,i}$$

$$e_{-\lambda_i} = e_{i+1,1} - e_{1,i+n+1} .$$

A system $\Pi$ of simple roots is:

$$\alpha_1 = \lambda_1 - \lambda_2$$

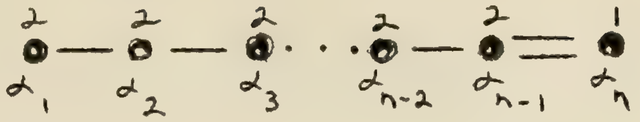
$$\alpha_2 = \lambda_2 - \lambda_3$$

$$\vdots$$

$$\alpha_{n-1} = \lambda_{n-1} - \lambda_n$$

$$\alpha_n = \lambda_n .$$

The Dynkin diagram is $B_n$ --



Thus the diagram $B_n$ is the Dynkin diagram of a simple Lie algebra; viz., $\mathcal{SO}(2n+1,K)$. As $B_1 = A_1$ we consider only $B_n$ with $n \geq 2$.

## The simple Lie algebras of type $C_n$

Let $\mathcal{Sp}(n,K)$ be the (sympletic) algebra of transformations of trace = 0 keeping invariant a skew-symmetric bilinear form in 2n variables. We ~~make~~ take as matrix of the bilinear form

$$\tau = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}.$$

Then $\mathcal{Sp}(n,K)$ is seen to consist of matrices such that $X\tau + {}^t\tau X = 0$; i. e.,

$$X = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & -{}^t a_{11} \end{pmatrix} \qquad \text{with } a_{12} = {}^t a_{12} \text{ and } a_{21} = {}^t a_{21}.$$

$\mathfrak{H}$ consists of the diagonal matrices. We have root vectors:

$$e_{\lambda_j - \lambda_i} = e_{i,j} - e_{j+n,i+1}$$

$$e_{-\lambda_i-\lambda_j} = e_{i,j+n} - e_{j,i+n} \qquad i < j$$

$$e_{\lambda_i + \lambda_j} = e_{i+n,j} + e_{j+n,i} \qquad i < j$$

$$e_{2\lambda_i} = e_{i,i+n}$$

$$e_{2\lambda_i} = e_{i+n,i} \quad .$$

For a system $\Pi$ of simple roots we take

$$\alpha_1 = \lambda_1 - \lambda_2$$

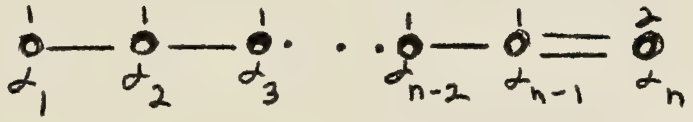
$$\alpha_2 = \lambda_2 - \lambda_3$$

$$\vdots$$

$$\alpha_{n-1} = \lambda_{n-1} - \lambda_n$$

$$\alpha_n = 2\lambda_n \quad .$$

The Dynkin diagram is, in this case, $C_n$



Thus the diagram $C_n$ is the Dynkin diagram of a simple Lie algebra; viz., $\mathcal{Sp}(n,K)$. As $C_1 = B_1 = A_1$ and $C_2 = B_2$ we take $C_n$ for $n \geq 3$.

The simple Lie algebras of type $D_n$

We take $\mathcal{SO}(2n,K)$, the Lie algebra of the special orthogonal group in $2n$ variables. $\mathfrak{H}$ is again the diagonal matrices in $\mathcal{SO}(2n,K)$ and the roots are

$$\lambda_i - \lambda_j$$

$$\pm(\lambda_i + \lambda_j) \quad .$$

As a system $\Pi$ we take,

$$\alpha_1 = \lambda_1 - \lambda_2$$

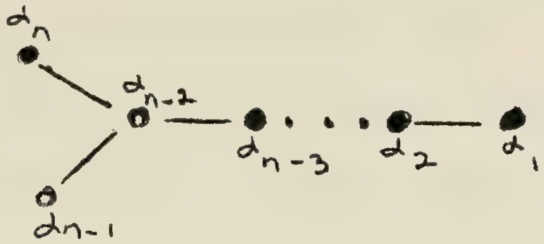
$$\alpha_2 = \lambda_2 - \lambda_3$$

$$\vdots$$

$$\alpha_{n-1} = \lambda_{n-1} - \lambda_n$$

$$\alpha_n = \lambda_{n-1} + \lambda_n$$

The Dynkin diagram has the form $D_n$ --



Thus the diagram $D_n$ is the Dynkin diagram of a simple Lie algebra; viz., $\mathscr{SO}(2n,K)$. As $D_1 = C_1 = B_1 = A_1$, $D_2 = A_2$, $D_3 = A_3$, we take $D_n$ for $n \geq 4$

We state without proof here that there are simple Lie algebras over $K$ with Dynkin diagrams $G_2$, $F_4$, $E_6$, $E_7$ and $E_8$. It results that the diagrams $A_n(n \geq 1)$, $B_n(n \geq 2)$, $C_n(n \geq 3)$, $D_n(n \geq 4)$, $G_2$, $F_4$, $E_6$, $E_7$, $E_8$ are precisely the possible diagrams for simple Lie algebras.

# Lecture XXIX

Given a Lie algebra $\mathfrak{G}$, a Cartan subalgebra $\mathfrak{H}$, and an ordering in $\mathfrak{H}_0^*$ there corresponds a simple system of roots, $\Pi$, and a Dynkin diagram. We now show that given any other Cartan subalgebra $\mathfrak{H}'$ and ordering in $\mathfrak{H}_0'^*$ we get the same Dynkin diagram. We begin by proving the independence of the diagram on ordering in $\mathfrak{H}_0^*$.

## I. Independence of the Dynkin diagram on the system of simple roots

We know that if $\lambda$ is a weight of a representation of the three-dimensional simple Lie algebra generated by $\{e_\alpha, e_{-\alpha}, h_\alpha\}$ then $-\lambda$ is also a weight. Thus, for the root $\lambda$ of $\mathfrak{H}$ we consider the root

$$\lambda - \frac{2\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha = \lambda - r_{\lambda,\alpha}\alpha + q_{\lambda,\alpha}\alpha$$

where $\lambda + k\alpha$ is a root for $-r_{\lambda,\alpha} \leq k \leq q_{\lambda,\alpha}$. Let $S_\alpha$ be the map $S_\alpha : \mathfrak{H}^* \longrightarrow \mathfrak{H}^*$ defined by

$$S_\alpha(\lambda) = \lambda - \frac{2\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha.$$

Set $P_\alpha = \{h; h \varepsilon H \text{ and } \alpha(h) = 0\}$. Identifying $\mathfrak{H}$ with $\mathfrak{H}^*$ via the Killing form we see that $S_\alpha$ is just reflection in $P_\alpha$ with respect to the induced Killing metric on $\mathfrak{H}$. $S_\alpha$ is called the "Weyl reflection in $P_\alpha$". We define the group $W$ generated by the $S_\alpha$ the Weyl group on $\mathfrak{H}$. Since $S_\alpha$ permutes the roots and $\langle S_\alpha(\lambda), S_\alpha(\lambda')\rangle = \langle\lambda,\lambda'\rangle$, each element of $W$ is induced by an automorphism of $\mathfrak{G}$. More specifically

Proposition Each element of $W$ is induced by an inner automorphism of $\mathfrak{G}$.

Proof Consider the 3-dimensional simple Lie algebra $\{e_\alpha, e_{-\alpha}, h_\alpha\}$. The Weyl reflection is, here, $h_\alpha \longrightarrow -h_\alpha$. Denoting $\langle\alpha,\alpha\rangle$ by $a$ we get:

$$[e_\alpha - e_{-\alpha}, \sqrt{2/a}\ h_\alpha] = -a\sqrt{2/a}\ (e_\alpha + e_{-\alpha})$$

$$[e_\alpha - e_{-\alpha}, e_\alpha + e_{-\alpha}] = 2h_\alpha = \sqrt{2a}\ \sqrt{2/a}\ h_\alpha$$

Let $Y$ be the restriction of $\mathrm{ad}(e_\alpha - e_{-\alpha})$ to the 2-dimensional subspace generated by $H_\alpha$ and $e_\alpha + e_{-\alpha}$. The matrix of $Y$ with respect to the basis $\{\sqrt{2/a}\ h_\alpha, e_\alpha + e_{-\alpha}\}$ has been computed above to be:

$$\begin{pmatrix} 0 & \sqrt{2a} \\ -\sqrt{2a} & 0 \end{pmatrix}$$

which we may write $\sqrt{2a}\ Z$ where $Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We have

$$e^{tY} = \sum_{p=0}^{\infty} \frac{(tY)^{2p}}{2p!} + \sum_{p=0}^{\infty} \frac{(tY)^{2p+1}}{(2p+1)!} .$$

Bearing in mind that $Z^2 = -I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ we write

$$(tY)^{2p} = (\sqrt{2a}\ tZ)^{2p} = (\sqrt{2a}\ t)^{2p}(-I)^p$$

$$(tY)^{2p+1} = (\sqrt{2a}\ tZ)^{2p}(\sqrt{2a}\ tZ) = (\sqrt{2a}\ t)^{2p+1}(-I)^p Z$$

whence

$$e^{tY} = \cos(\sqrt{2a}\ t)I - (\sin\sqrt{2a}\ t)Z.$$

thus, if we choose $t = \pi/\sqrt{2a}$ we have:

$$\exp \pi/\sqrt{2a}\ Y = -I.$$

Setting $g = \exp \operatorname{ad} \pi/\sqrt{2a}(e_\alpha - e_{-\alpha})$ we see

$$gh_\alpha = -h_\alpha$$

$$gh = h \quad \text{for} \quad h \in P_\alpha$$

The result follows for the general case.

Definition Let $\mathcal{R}$ be the set of non-zero roots on $\mathfrak{H}$. By a Weyl chamber in $\mathfrak{H}_0$ we mean a maximal convex subset of $\mathfrak{H}_0 - \bigcup_{\alpha \in \mathcal{R}} P_\alpha$. It is a maximal subset of $\mathfrak{H}_0$ on which no root vanishes or changes sign.

The Weyl group $W$ permutes the chambers. What's more -

Lemma 1 $W$ permutes the Weyl chambers transitively.

Proof Let $C_0$ and $C_1$ be Weyl chambers with $h_0 \in C_0$ and $h_1 \in C_1$. It suffices to prove that there is an $S \in W$ such that $h_0$ and $sh_1$ are on the same side of each plane $P_\alpha$.

Choose $S$ in $W$ so as to mimimize the Killing distance $|h_0 - sh_1|$. If $h_0$ and $sh_1$ were on opposite sides of $P_\alpha$ then the straight line joining them would intersect $P_\alpha$ and we would have $s_\alpha sh_1$ nearer to $h_0$ -- a contradiction. Thus we have $sh_1 \in C_0$ and $sC_1 = C_0$.

Lemma 2 There is a canonical one-to-one correspondence between Weyl chambers and systems of simple roots.

Proof Given a system of simple roots $\Pi = \{\alpha_1, \alpha_2, \cdots, \alpha_\ell\}$

then $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ are linearly independent and each root $\alpha$ can be written

$$\alpha = \pm \sum_{i=1}^{\ell} m_i \alpha_i \ ; m_i \geq 0$$

Set $C(\Pi) = \{h ; \alpha_i(h) > 0$ for $i = 1, 2, \ldots, \ell\}$. $C(\Pi)$ is a Weyl chamber.

Conversely, given a chamber $C$, let $\Pi(C)$ consists of all roots which are positive on $C$ but not a sum of two such roots. For any element $s \varepsilon W$ we have

$$\Pi(sC) = s^*(\Pi(C))$$

since $\langle s^*\lambda, sh\rangle = \langle\lambda, h\rangle$. Thus, to prove that $\Pi$ is a system of simple roots - it suffices to prove that $\Pi(sC)$ is a system of simple roots for some $s \varepsilon W$; i. e., for some chamber $C$. But $\Pi(C(\Pi)) = \Pi$. Thus $\Pi(C)$ is a system of simple roots for any chamber $C$ and also $C \longrightarrow \Pi(C)$ is a bijection.

As a corollary to Lemmas 1 and 2 it follows immediately that any two systems of simple roots are isomorphic under the Weyl group -- in particular, any two systems of simple roots yield the same Dynkin diagram.

## II. Independence of the Dynkin diagram on the choice of H.

Let $\mathcal{H}$ and $\mathcal{H}'$ be Cartan subalgebras of $\mathcal{G}$. Let $\mathcal{R}$ and $\mathcal{R}'$ be the sets of roots on $\mathcal{H}$ and $\mathcal{H}'$ respectively. We define

$$P(\lambda) = P(\lambda,X) = \det(\lambda I - \operatorname{ad} X) = \sum_{k=0}^{n} B_k\lambda^k$$

$$\bar{P}(\lambda) = P(\lambda,h) = \det(\lambda I - \operatorname{ad} h) = \sum_{k=0}^{n} \bar{B}_k\lambda^k \quad \text{for} \quad h \in \mathfrak{H}$$

where $n = \dim \mathfrak{G}$. $P(\lambda)$ is a polynomial in the indeterminant $\lambda$ with coefficients in the field $K(B_1,B_2,\dots,B_n)$ where $B_k$ is a polynomial function on $\mathfrak{G}$. $\bar{P}(\lambda)$ has its coefficients in the field $K(\bar{B}_1,\bar{B}_2,\dots,\bar{B}_n)$. We assume, for the proof we shall give, that $K$ is the field of complex numbers.

Definition The adjoint group $G$ of the Lie algebra $\mathfrak{G}$ (or the group of inner automorphisms) is the linear group where Lie algebra is $\operatorname{ad}\mathfrak{G}$.

Recall that $P(\lambda,g(x) = P(\lambda,x)$ for all $g \in G$, so that $B_k(gx) = B_k(x)$ for all $g \in G$.

Lemma 3 The restriction homomorphism is an isomorphism of $K(B_1,\dots,B_n)$ onto $K\langle\bar{B}_1 \dots,\bar{B}_n\rangle$

Proof Consider the polynomial mapping $\rho: G \times \mathfrak{H} \longrightarrow \mathfrak{G}$ defined by $\rho(g,h) = g(h)$ for $g \in G$ and $h \in \mathfrak{H}$. As a polynomial mapping $\rho$ is certainly analytic, and in order to prove that the image contains a $\mathfrak{G}$-neighborhood of an element $h \in \mathfrak{H}$ it is sufficient to prove that the differential of the map $\rho$ has rank equal to the dimension of $\mathfrak{G}$. Now

$$d\rho_{(1,h)}(h) = h$$

$$d\rho_{(1,h)}(x) = \operatorname{ad} x(h) = [x,h].$$

Thus $d\varphi$ of the tangent space at $(1,h) = H + \operatorname{ad} h(\mathfrak{G})$ $- \mathfrak{G}$ if $h$ is regular

To see that the restriction homomorphism is one-to-one we assume a polynomial relation $f(\bar{B}_1, \ldots, \bar{B}_n) = 0$. Then $f(B_1(h), \ldots, B_n(h)) = 0$ for all $h \varepsilon \mathfrak{H}$. Thus $f(B_1(gh), \ldots, B_n(gh)) = 0$ for all $h \varepsilon \mathfrak{H}$ and $g \varepsilon G$. From the paragraph above we conclude that $f(B_1(x), \ldots, B_n(x)) = 0$ for all $x$ in some $\mathfrak{G}$-neighborhood of the regular element $h$. It follows that $f(B_1, \ldots, B_n) = 0$

$\mathcal{R}$ is the set of roots of $\bar{P}(\lambda)$ since

$$\bar{P}(\lambda) = \prod_{\alpha \varepsilon \mathcal{R}} (\lambda - \alpha).$$

$\mathcal{R}$ can be identified with the roots of $P(\lambda)$ over the field $K(B_1, \ldots, B_n)$ by virtue of Lemma 3. There results a one-to-one correspondence between roots $\mathcal{R}$ and $\mathcal{R}'$ taking sums into sums, whence the euclidean spaces $\mathfrak{H}$ and $\mathfrak{H}'$ are isometric via some map say $\varphi$. Let $\Pi$ be a simple system of roots for $\mathfrak{H}$. Then $\varphi(\Pi)$ is a simple system for $\mathfrak{H}'$ and we see that the Dynkin diagrams are the same for any $\mathfrak{H}$ and $\mathfrak{H}'$.

Remark For an algebraically closed ground field $K$ the image of $p: G \times \mathfrak{H} \longrightarrow \mathfrak{G}$ is a $\mathfrak{G}$-neighborhood in the Zariski - topology of $\mathfrak{G}$. Since any two Zariski - neighborhoods of $\mathfrak{G}$ intersect in non-empty Zariski - neighborhood, we can find for any two Cartan subalgebras $\mathfrak{H}_1$ and $\mathfrak{H}_2$ a regular element $h$ conjugate to an element of $\mathfrak{H}_1$ and to an element of $\mathfrak{H}_2$. It follows at once that $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are conjugate under an element of $G$.

# LECTURE XXX

In this lecture we discuss some facts about algebraic linear groups.

Let $V$ be a finite dimensional vector space (with dimension $V = n$) over the field $k$. Let $E = \mathrm{Hom}_k(V,V)$ and denote by $GL(V)$ the set of invertible elements in $E$. A subgroup $G$ of $GL(V)$ is called algebraic in case there exists a set $\mathcal{P}$ of polynomial functions on the linear space $E$ such that $G = \{g; g \in GL(V)$ and $\rho(g) = 0$ for all $\rho \in \mathcal{P}\}$.

For any $g \in GL(V)$ we define a map $L(g): E \longrightarrow E$ by the formula $L(g)e = g \cdot e$. Denote by $L_{r,s}(g)$ the canonical operation of $L(g)$ on the tensor space $E_{r,s}$ over $E$. A tensor $t \in E_{r,s}$ is called a semi-invariant of $G$ whenever $L_{r,s}(g)t = \lambda(g)t$ with $\lambda(g) \in k$ for all $g \in G$. When this is the case $\lambda(g_1 g_2) = \lambda(g_1)\lambda(g_2)$ and $\lambda$ is called the weight of the semi-invariant $t$.

Theorem: An algebraic linear group $G$ in $GL(V)$ is defined by a finite set of semi-invariants of equal weights.

Proof: Let $\mathcal{P}$ be the ideal of all polynimial functions on $E$ which vanish on $G$. An element $x \in GL(V)$ is in $G$ if and only if $\mathcal{P}$ is stable under $L(x)$ -- i.e., $\mathcal{P} \circ L(x) = \mathcal{P}$. To see this, first assume $\mathcal{P} \circ L(x) = \mathcal{P}$; then

$$\rho(xe) = 0 \text{ for all } e \in G \text{ and } \rho \in \mathcal{P},$$

in particular

$$\rho(x1) = \rho(x) = 0 \text{ for all } \rho \in \mathcal{P}$$

and $x \in G$. Conversely, if $x \in G$ and $\rho \in \mathfrak{P}$ we have

$$(\rho \circ L(x))(e) - \rho(xe) = 0 \quad \text{for} \quad e \in G ,$$

i,e., $\mathfrak{P} \circ L(x) \subset \mathfrak{P}$, and since $\rho(e) = (\rho \circ L(x^{-1}) \circ L(x))(e)$ and (by the same argument) $\rho \circ L(x^{-1}) \subset \mathfrak{P}$ we see $\mathfrak{P} \subset \mathfrak{P} \circ L(x)$.

Now $\mathfrak{P}$ is an ideal in a polynomial ring in $n^2$-variables over the field $k$ and, by the Hilbert basis theorem, it is finitely generated. Set $\mathfrak{P}_\ell$ = the set of elements in $\mathfrak{P}$ of degree $\leq \ell$. Then $\mathfrak{P}_\ell$ generates $\mathfrak{P}$ for some $\ell$. We choose such an $\ell$ and we have from the above considerations:

$$x \in G \text{ if and only if } \mathfrak{P}_\ell \text{ is stable under } L(x) .$$

Set $\Lambda$ equal to the exterior algebra built over the space of all polynomial functions on $E$ of degree $\leq \ell$. Set $\hat{\mathfrak{P}}_\ell$ equal to the canonical 1-dimensional subspace of $\Lambda$ corresponding to $\mathfrak{P}_\ell$, whence

$$(*) \quad x \in G \text{ if and only if } L(x) \text{ keeps } \hat{\mathfrak{P}}_\ell \text{ stable} .$$

Now the elements of $\Lambda$ can be thought of as alternating tensors and $\Lambda$ is identified with a subspace of the direct sum of tensor spaces $\sum_{i=1}^{N} T_i$. Let $t$ be a non-zero element in $\hat{\mathfrak{P}}_\ell$ and $\hat{x}$ (respectively $x_i$) be the canonical operation of $x$ on $\Lambda$ (respectively $T_i$). We have

$$t = t_1 + t_2 + \dots + t_N$$

$$\hat{x}t = x_1 t_1 + x_2 t_2 + \dots + x_N t_N$$

and the condition

$$x_i t_i = \lambda t_i \qquad (i = i,2,\ldots,N)$$

is equivalent (by (*)) to the condition that $x \varepsilon G$

Conversely, we can show that the set of all elements $x \varepsilon GL(V)$ such that $x$ admits $t_1,t_2,\ldots,t_N$ as eigenvectors of the same weight, $\lambda(x)$, forms an algebraic group.

Corollary: Let $x \varepsilon G$ and let $x = S + N$ be its Jordan normal form decomposition. Then:

a) $S$ and $(1 + S^{-1}N)$ are in $G$

b) Set $U = (1 + S^{-1}N)$. Then $U$ is unipotent; i.e., all its eigenvalues are = 1. The endomorphism

$$\log U = \sum_{\ell=1}^{\infty} (-1)^{\ell+1} \frac{(U-I)^{\ell}}{\ell}$$

is well-defined, since $(U-I)^{\ell} = 0$ for sufficiently large. If we set

$$U^c = e^{c \log U}$$

then $U^c$ lies in $G$ for all $c \varepsilon k$.

c) Assume $k = R$ = real numbers. Assume that the eigenvalues of $S$ are all positive. Define

$$\text{Log } S = \sum_{\ell=1}^{\infty} (-1)^{\ell+1} \frac{(\frac{1}{N}S-1)^{\ell}}{\ell} + \log N\cdot 1$$

where $N$ is a number greater than 1/2 the maximum eigenvalue of $S$ (this definition is independent of $N$). Then $S^t = e^{t \log S}$ is in $G$ for every $t \varepsilon R$.

d) Assume $k = R$ = real numbers. Write

$$S = TR$$

where $TR = RT$, all eigenvalues of $T$ are positive, all eigenvalues of $R$ are of modulus $= 1$, and $T$ and $R$ are both semi-simple. (We call $T$ the absolute value of $S$ and $R$ the rotational part of $S$). Then $T$ and $R$ are in $G$.

Proof: We give proofs of a) and b). Theproofs of c) and d) are quite similar.

a) Let $t_1, \ldots, t_N$ be the semi-invariants defining $G$. Then $L_{r,s}(x)t_i = \lambda t_i$ for $i = 1,2,\ldots,N$. It follows that

$$L_{r,s}(S)\, t_i = f(\lambda)t_i$$

where $f$ is a polynomial such that $S = f(x)$ because the tensor representation takes the Jordan decomposition into the Jordan decomposition. It follows that $S$ is in $G$ and, therefore, $(1 + S^{-1}N) = S^{-1}x \in G$.

b) We give the proof for $k$ = real numbers. First observe:

$$L_{r,s}(U)t_i = L_{r,s}(S^{-1})L_{r,s}(x)t_i = t_i$$

What we want to show is:

$$L_{r,s}(U^c) = L_{r,s}(U)^c \quad \text{for} \quad c \in R$$

This follows from the formula

$$\exp c[dL_{r,s}(X)] = L_{r,s}(\exp cX)$$

by setting $X = \log U$. We leave the above formula as an exercise.

# Lecture XXXI

We prove here some facts about "compact" Lie algebras.

Definition the real Lie algebra $\mathcal{G}$ is called compact provided that the Killing form is negative definite (a fortieri $\mathcal{G}$ is semi-simple).

We have the following characterization of compact Lie algebras:

Theorem 1 A semi-simple Lie algebra $\mathcal{G}$ over the field R of real numbers is compact if and only if its adjoint group $\mathrm{Ad}\,\mathcal{G}$ is compact.

For the proof we shall require the following fact:

Lemma. Every derivation D of the semi-simple Lie algebra $\mathcal{G}$ over the field k is an inner-derivation; i.e., $D = \mathrm{ad}\, X_0$ for some $X_0 \in \mathcal{G}$.

Proof.

We have, for any X and Y in $\mathcal{G}$

$$D[X,Y] = [DX,Y] + [X,DY];$$

i.e.,

$$D \circ \mathrm{ad}\, X = \mathrm{ad}\, DX + (\mathrm{ad}\, X)\circ D, \quad \text{or}$$

$$\mathrm{ad}\, DX = [D,\mathrm{ad}\, X] .$$

Let $\mathcal{E} = \mathrm{Hom}_k(\mathcal{G},\mathcal{G})$. Then $\dim \mathcal{E} = (\dim \mathcal{G})^2$ and

$$\mathcal{E} = \mathcal{E}_0 + \mathrm{ad}\,\mathcal{G} \quad \text{(direct)}$$

where $\mathcal{E}_0 = \{Y;\ Y \in \mathcal{E} \text{ and } \mathrm{tr}\, Y\, \mathrm{ad}\,\mathcal{G} = 0\}$.

Furthermore, $[\mathrm{ad}\,\mathcal{G}, \mathcal{E}_0] \subset \mathcal{E}_0$.

Set $D = D_0 + \mathrm{ad}\, X_0$ with $D_0 \in \mathcal{E}_0$ and $X_0 \in \mathcal{G}$.

We have $[D_0, \mathrm{ad}\, X] \in \mathcal{E}_0$ for all $X \in \mathcal{G}$ so that

$$\mathrm{ad}\, DX = [D,\mathrm{ad}\, X] = [D_0,\mathrm{ad}\, X] + [\mathrm{ad}\, X_0,\mathrm{ad}\, X]$$
$$= [D_0,\mathrm{ad}\, X] + \mathrm{ad}[X_0,X]$$

It follows that $[D_0, \mathrm{ad}\, X] = 0$ and

$$\mathrm{ad}\, DX = \mathrm{ad}\, [X_0, X],$$

whence $DX = [X_0, X]$ for all $X \in \mathcal{G}$ (since the center of $\mathcal{G}$ is trivial).

Proof of Theorem 1:

I. As we have seen before, the derivation algebra of $\mathcal{G}$ is the Lie algebra of $A(\mathcal{G})$, the group of automorphisms of the Lie algebra $\mathcal{G}$. Thus if $\mathcal{G}$ is semi-simple, the Lie algebra of $A(\mathcal{G})$ is $\mathrm{ad}\, \mathcal{G}$. The group $A(\mathcal{G})$ is an algebraic linear group, being the family of endomorphisms preserving the structure constants of $\mathcal{G}$. Thus $\mathrm{Ad}\, \mathcal{G}$, the adjoint group of $\mathcal{G}$ (~~the analytic subgroup whose~~ Lie algebra is $\mathrm{ad}\, \mathcal{G}$) is the connected component of the identity in $A(\mathcal{G})$.

Since $A(\mathcal{G})$ is closed in the full linear group, so is $\mathrm{Ad}(\mathcal{G})$. Now assume $\mathcal{G}$ is compact. Then $\mathrm{Ad}(\mathcal{G})$ is a closed subgroup of the full linear group which preserves some negative definite form. It follows that $\mathrm{Ad}(\mathcal{G})$ is compact.

II. Conversely, if $\mathrm{Ad}\, \mathcal{G}$ is compact and $\mathcal{G}$ is semi-simple then $\mathrm{Ad}\, \mathcal{G}$ is a bounded group of matrices and for each $X \in \mathcal{G}$ $\mathrm{ad}\, X$ cannot have any eigenvalues of non-zero real part $r$ -- otherwise $\exp \mathrm{ad}\, tX$ has an eigenvalue of absolute value $e^{tr}$, which is unbounded. It follows that $\mathrm{tr}(\mathrm{ad}\, X)^2 \leq 0$ and the Killing form is negative semi-definite. Since the Killing form is non-degenerate it is negative definite.

**Definition.** Let $\mathcal{G}$ be a Lie algebra over the field $k$ and let $K$ be a field containing $k$. The injection $i_k: k \longrightarrow K$ defines a canonical injection $i_{\mathcal{G}} \otimes i_k$: $\mathcal{G} = \mathcal{G} \otimes_k k \longrightarrow \mathcal{G} \otimes_k K$ where $i_{\mathcal{G}}$ denotes the identity mapping of $\mathcal{G}$ onto itself. The vector space $\mathcal{G} \otimes_k K$ can be given a Lie algebra structure by taking any base $\{x_1, \ldots, x_n\}$ and structure constants $c_{ij}^k$ $(i,j,k = 1,2,\ldots,n)$ for $\mathcal{G}$ as structure for $\mathcal{G} \otimes_k K$.

Given a Lie algebra $\mathcal{G}_K$ over the field $K$, we mean by a $k$-form of $\mathcal{G}_K$ a subset $\mathcal{G}$ of $\mathcal{G}_K$ such that $\mathcal{G}$ is a vector space over $k$ and there exists a $K$-isomorphism $\alpha$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "\mathrm{id}"] \arrow[d] & \mathcal{G} \arrow[d, "\mathrm{id}"] \\
\mathcal{G} \otimes_k K \arrow[r, "\alpha"] & \mathcal{G}_K
\end{tikzcd}$$

We have for Lie algebras $\mathcal{G}$ over the complex numbers,

**Theorem 2.** A semi-simple Lie algebra $\mathcal{G}$ over the field $C$ of complex numbers has a compact real form.

**Proof.** Let $\mathcal{H}$ be a Cartan subalgebra of $\mathcal{G}$ and let $e'_\alpha \in \mathcal{G}_\alpha$, $e'_{-\alpha} \in \mathcal{G}_{-\alpha}$, $h'_\alpha \in \mathcal{H}$ $(\alpha \neq 0)$ be such that

(i) $[e'_\alpha, e'_{-\alpha}] = h'_\alpha$

(ii) $B(e'_\alpha, e'_{-\alpha}) = 1$ .

Let $\sigma : \mathcal{G} \longrightarrow \mathcal{G}$ be the involution defined by $\sigma(e'_\alpha) = e'_{-\alpha}$. Then $\sigma^2 = 1$ on $\mathcal{G}$ and $\sigma(\mathcal{G}_\alpha) = \mathcal{G}_{-\alpha}$ for $\alpha \neq 0$ and $\sigma(\mathcal{H}) = \mathcal{H}$. It follows that $\eta_\alpha = B(e'_\alpha, e'^{\sigma}_\alpha)$ is different from zero and we may solve the equation:

$$\lambda_\alpha^2 \eta_\alpha = -1$$

Setting $e_\alpha = \lambda_\alpha e'_\alpha$ we get

$$B(e_\alpha, e_\alpha^\sigma) = -1$$

and if we take $e_{-\alpha} = e_\alpha^\sigma$ for positive roots $\alpha$ we have for all non-zero roots $\alpha$

$$B(e_\alpha, e_{-\alpha}) = -1.$$

As a basis for $\mathcal{G}$ we take $\{h_1, h_2, \ldots, h_\ell, e_\alpha, e_{-\alpha}\}$ where $\alpha$ runs over all non-zero roots and $\ell$ is the rank of $\mathcal{G}$.

Let $\alpha$ and $\beta$ be non-zero roots with $\alpha \neq \pm\beta$ and let $[e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta}$. An elementary computation yields that $N^2_{\alpha,\beta} \geq 0$ and since $\alpha(h_i)$ is rational for $i = 1, 2, \ldots, \ell$ the basis above has real structure constants and the set $\mathcal{G}'$ of all real linear combinations of this basis is a real Lie algebra whose complexification $\mathcal{G}' \otimes_R C$ is $\mathcal{G}$.

Let $\mathcal{G}_K$ be the real Lie algebra consisting of all real linear combinations of the elements $\sqrt{-1}\, h_i$ $(i = 1, 2, \ldots, \ell)$, $e_\alpha + e_{-\alpha}$, and $\sqrt{-1}(e_\alpha - e_{-\alpha})$ for $\alpha$ a positive root. We leave

as an exercise to the reader to show that $\mathcal{G}_K$ is a compact real form of $\mathcal{G}$. We may also write $\mathcal{G}_K$, above, as

$$\mathcal{G}_K = \sqrt{-1}\,\mathcal{H}_0 + \sum_{\alpha \neq 0} (z_\alpha e_\alpha - \bar{z}_\alpha e_{-\alpha})$$

where $\mathcal{H}_0$ is the real span of the $\{h_i\}$ and $z_\alpha$ is an arbitrary complex number.

**Remark.** Let $\mathcal{G}$ be a complex, semi-simple Lie algebra and $\mathcal{G}_K$ an arbitrary compact form. We can pick a basis for $\mathcal{G}$ of elements $\{h_i\}_{i=1,2,\ldots,\ell}$ and root vectors $\{X_\alpha, X_{-\alpha}\}_{\alpha \neq 0}$ such that $\mathcal{G}_K$ can be written

$$\mathcal{G}_K = \sum_{i=1}^{\ell} R\sqrt{-1}h_i + \sum_{\alpha \neq 0} R(X_\alpha - X_{-\alpha}) + \sum_{\alpha \neq 0} R\sqrt{-1}(X_\alpha + X_{-\alpha}).$$

This results from the fact that for any two compact real forms $\mathcal{G}_K$ and $\mathcal{G}'_K$ of $\mathcal{G}$ there is an automorphism $\sigma$ (over C) of $\mathcal{G}$ such that $\sigma\mathcal{G}_K = \mathcal{G}'_K$ (this fact can be proved in a manner similar to the proof of Theorem 3 below). As we already have a basis of the prescribed form for one compact form $\mathcal{G}_K$ we see that $\mathcal{G}'_K$ can be written:

$$\mathcal{G}'_K = \sum_{i=1}^{\ell} R\sqrt{-1}\sigma(h_i) + \sum_{\alpha \neq 0} R(\sigma X_\alpha - \sigma X_{-\alpha}) + \sum_{\alpha \neq 0} R\sqrt{-1}(\sigma X_\alpha + \sigma X_{-\alpha})$$

We now prove a sharper form of Theorem 2.

**Theorem 3.** Let $\mathcal{G}$ be a semi-simple Lie algebra over the field R of real numbers. Let $\mathcal{G}_C$ denote the complexification of $\mathcal{G}$ and $\sigma$

the complex conjugation of $\mathcal{G}_C$ with respect to $\mathcal{G}$.

Then there is a compact real form of $\mathcal{G}_C$ stable under $\sigma$.

Proof. Let $\mathcal{O}$ be a real form of $\mathcal{G}_C$ and let $\tau$ denote the conjugation of $\mathcal{G}_C$ with respect to $\mathcal{O}$. Then $\mathcal{O}$ is stable under $\sigma$ just in case $\sigma\tau$ and $\tau\sigma$ coincide with $\sigma$ on $\mathcal{O}$. Now $\sigma\tau$ and $\tau\sigma$, being products of conjugations, are C-linear maps (in fact, they are automorphisms of $\mathcal{G}_C$ as a Lie algebra over C) and so they are determined by their values on $\mathcal{O}$ -- a real form of $\mathcal{G}_C$. Thus we have $\mathcal{O}$ stable under $\sigma$ if and only if $\sigma\tau = \tau\sigma$.

Now let $\mathcal{O}$ be a compact real form of $\mathcal{G}_C$ and let $\tau$ be complex conjugation with respect to $\mathcal{O}$. We will demonstrate the existence of an automorphism Q in Ad $\mathcal{G}_C$ such that $\sigma(Q\tau Q^{-1})\sigma^{-1} = Q\tau Q^{-1}$. Then $Q\mathcal{O}$ is the desired form and $Q\tau Q^{-1}$ is the conjugation of $\mathcal{G}_C$ with respect to $Q\mathcal{O}$.

Set $N = \sigma\tau$. Then N is an automorphism of $\mathcal{G}_C$ (as Lie algebra). Set $H(X,Y) = B(X,\tau Y)$ where X and Y are in $\mathcal{G}_C$ and B is the Killing form. H(X,Y) is an Hermitian form and is negative definite, since it is negative definite on $\mathcal{O}$ ($\mathcal{O}$ is compact). Furthermore,

$$\begin{aligned} H(NX,Y) &= B(NX,\tau Y) \\ &= B(X,N^{-1}\tau Y) \\ &= B(X,\tau\sigma\tau Y) \\ &= H(X,NY) \ , \end{aligned}$$

and we see that $N$ is self-adjoint with respect to the definite Hermitian form $H$. Thus $N$ has real eigenvalues and is semi-simple. We set $P = (N^2)^{1/2}$. Then $P^t$ is in Ad $\mathscr{G}_C$ for all real $t$ (it is in Aut($\mathscr{G}_C$,$\mathscr{G}_C$) since Aut($\mathscr{G}_C\mathscr{G}_C$) is algebraic). Thus $P^{1/2}$ is in Ad $\mathscr{G}_C$.

We have

1) $\sigma P^2\sigma^{-1} = \sigma\sigma\tau\sigma\tau\sigma = P^{-2}$

2) $\sigma P^t\sigma^{-1} = P^{-t}$ for all real $t$, as may be seen by comparing the power series expansions for the two sides.

Now

$$\begin{aligned}\sigma(P^{1/2}\tau P^{-1/2})\sigma^{-1} &= \sigma P^{1/2}(\sigma\sigma)\tau(\tau\tau)(\sigma\sigma)P^{-1/2}\sigma \\ &= (\sigma P^{1/2}\sigma)(\sigma\tau)\tau(\sigma\tau)^{-1}(\sigma P^{-1/2}\sigma) \\ &= P^{-1/2}N\tau N^{-1}P^{1/2}\end{aligned}$$

We will show $P^{1/2}\tau P^{-1/2} = P^{-1/2}N\tau N^{-1}P^{1/2}$; i.e.

$$P\tau P^{-1} = N\tau N^{-1} .$$

This amounts to

$$N^{-1}P\tau P^{-1}N = \tau.$$

Since

1) $\sigma P^t\sigma^{-1} = P^{-t}$

2) $\tau P^t\tau^{-1} = P^{-t}$

3) $NP^tN^{-1} = \sigma(\tau P^t\tau^{-1})\sigma^{-1} = \sigma P^{-t}\sigma^{-1} = P^t$,

the above equation becomes --

$$N^{-1}PP\tau(\sigma\tau) = \tau$$

$$N^{-1}P^2N^{-1}\tau = \tau$$

or $$P^2\tau = N^2\tau .$$

But this last equation is known to be true since $P^2 = N^2$. We set $Q = P^{1/2}$ and we have

$$\sigma(Q\tau Q^{-1})\sigma^{-1} = Q\tau Q^{-1}.$$

In this lecture we examine the topology of $GL(n,R)$.

We **write** $GL(n,R)$ for the group of non-singular $n \times n$ matrices with real coefficients. The orthogonal subgroup is denoted $O(n,R)$. Let $\mathcal{S}$ be the set of all real symmetric $n \times n$ matrices and $P$ the set of positive definite symmetric real $n \times n$ matrices. We topologize $\mathcal{S}$ as a linear space and give $P$ the relative topology. The eigenvalues of elements of $P$ are all positive (this is easily seen from the fact that for $p \in P$ there exists $\sigma \in O(n,R)$ with ${}^t\sigma p \sigma$ equal to a diagonal matrix).

Lemma 1: $\exp : \mathcal{S} \longrightarrow P$ is a homeomorphism of $\mathcal{S}$ onto $P$.

Proof: A) The map exp above is surjective.

For $p \in P$ let $d = \text{diag}(\lambda_1,\ldots,\lambda_n)$ be the diagonal matrix whose non-zero entries $\lambda_i > 0$ are the eigenvalues of $p$. We may write

$$p = gdg^{-1} = gd\,{}^tg$$

where $g$ is orthogonal. Setting $X = g(\text{diag}(\log\lambda_1,\ldots,\log\lambda_n))g^{-1}$ we have:

$$p = \exp X$$

b) The map exp above is injective

Let $P \ni p = \exp X_1 = \exp X_2$ with $X_1, X_2 \in \mathcal{S}$. Both $X_1$ and $X_2$ are diagonalizable. No generality is lost in assuming $X_1 = \text{diag}(\lambda_1,\ldots,\lambda_n)$, $X_2 = g\ \text{diag}(\lambda_1,\ldots,\lambda_n)g^{-1}$. From the equation

$$\exp \text{diag}(\lambda_1,\ldots,\lambda_n) = p = g(\exp \text{diag}(\lambda_1,\ldots,\lambda_n))g^{-1}$$

we have that $g$ commutes with $\exp \text{diag}(\lambda_1,\ldots,\lambda_n)$.

We would like to show that $g$ commutes with $\mathrm{diag}(\lambda_1,\ldots,\lambda_n)$ itself. Write $g = (g_{ij})$ . We have seen

$$g_{ij}e^{\lambda_j} = e^{\lambda_i}g_{ij} \qquad \text{for all } i,j$$

whence; $g_{ij} = 0$ if $\lambda_i \neq \lambda_j$ . Then

$$g_{ij}\lambda_j = \lambda_i \; g_{ij} \quad \text{for all } i,j$$

and we have

$$g\ \mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n) = \mathrm{diag}(\lambda_1,\ldots,\lambda_n)g ,$$

i.e., $X_2 = X_1$ .

C) To see that exp is a homeomorphism we first observe that it is continuous (since $\mathcal{S}$ has the relative topology from $M(n,R)$ - the full matrix ring). $P$ is the set of all elements of $\mathcal{S}$ whose principal minors are positive and thus $P$ is open in $\mathcal{S}$ and is a manifold. As in general a one-to-one continuous map of one manifold onto another is a homeomorphism (by invariance of domain) we have the desired conclusion.

**Definition**: If $X \in \mathcal{S}$ and $\exp X \in P$ we call $X$ the logarithm of $p = \exp X$. We write $X = \log p$ and the map $p \longrightarrow p^{1/2} = \exp 1/2 \log p$ is a homeomorphism of $P$ onto $P$ .

**Theorem**: The map $\varphi: O(n,R) \times P \longrightarrow GL(n,R)$ defined by

$$\varphi(k,p) = k\cdot p$$

for $k \in O(n,R)$, $p \in P$ is a homeomorphism.

**Proof**: Let $g = k\cdot p$ . Then

$$p^2 = {}^t g g$$

from which we deduce $p = ({}^t gg)^{1/2}$ and $k = gp^{-1}$ ; in particular, $\varphi$ is one-to-one.

Given $g \in GL(n,R)$ we choose $p = ({}^t gg)^{1/2}$ and $k = gp^{-1}$ observing that tgg is symmetric and positive definite -- the positive definiteness follows from the formula

$$<tggv,v> = <gv,gv> \; > 0 \quad \text{if} \quad v \neq 0$$

where $< , >$ denotes the standard positive definite inner product in $R^n$ . Since ${}^t gg$ has all its eigenvalues positive $p = ({}^t gg)^{1/2}$ is defined and $\varphi$ is seen to be onto.

Observing that $p$ and $k$ are continuous functions of $g$ we see that $\varphi$ is a homeomorphism.

We have shown that $GL(n,R) = O(n,R) \times P$ topologically -- in particular, $GL(n,R)$ is topologically the product of a compact space with a euclidean space.

Lemma 2: Let $G$ be an algebraic subgroup of $GL(n,R)$ with $G = {}^t G = \{g; {}^t g \in G\}$ . Then

$$G = [G \cap O(n,R)] \cdot (G \cap P)$$

Proof: Let $g \in G$ . Then ${}^t gg \in G$ and is a diagonalizable element with positive eigenvalues. Hence $p = ({}^t gg)^{1/2}$ is in $G$ and $k = gp^{-1}$ is in $G$ . This, together with the Theorem above proves the lemma.

Remark: The result of the above lemma goes through when $G$ is the connected component of the identiey in an algebraic group.

**Definition**: We call k and p the Cartan components of g .

We consider, now a special case of the above situation. Let $\mathcal{G}$ be a real semi-simple Lie algebra. Let $\mathcal{G}_C$ be its complexification and $\sigma$ the complex conjugation of $\mathcal{G}_C$ with respect to $\mathcal{G}$. Let $\mathcal{G}_K$ be a compact real form of $\mathcal{G}_C$ stable under $\sigma$ . Then $\sqrt{-1}\,\mathcal{G}$ is the set of vectors in $\mathcal{G}_C$ corresponding to the eigenvalue $-1$ for p . Thus

$$\mathcal{G}_K = (\mathcal{G}_K \cap \mathcal{G}) + (\mathcal{G}_K \cap \sqrt{-1}\,\mathcal{G})$$

where the sum on the right is direct (as vector spaces over R). Let $\tau$ be the conjugation of $\mathcal{G}_C$ with respect to $\mathcal{G}_K$ . Then, by the same argument $(\tau\mathcal{G} = \mathcal{G})$

$$\mathcal{G} = (\mathcal{G} \cap \mathcal{G}_K) + (\mathcal{G} \cap \sqrt{-1}\,\mathcal{G}_K)$$

again a direct sum as vector spaces over R and $[\mathcal{G} \cap \mathcal{G}_K, \mathcal{G} \cap \sqrt{-1}\,\mathcal{G}_K] \subset \mathcal{G} \cap \sqrt{-1}\,\mathcal{G}_K$ .

**Lemma 3**: The Lie algebra ad $\mathcal{G}$ can be considered as a Lie algebra of matrices stable under transpose.

**Proof**: We must find on $\mathcal{G}$ a positive definite inner product (or a positive-definite quadratic form) with respect to which ad $\mathcal{G}$ is a self-adjoint family.

For $X \in \mathcal{G}$ we write $X = X + Y$ with $Y \in \mathcal{G} \cap \mathcal{G}_K$ and $Z \in \mathcal{G} \cap \sqrt{-1}\,\mathcal{G}_K$ . We define

$$H(X,X) = -B(Y,Y) + B(Z,Z) \quad .$$

H(X,X) is well defined and positive-definite.

For $x \in \text{ad}\,\mathcal{G}$ let ${}^t x$ be the adjoint of $x$ with respect to H ; i.e.,

$$H(xu,v) = H(u,{}^{t}xv)$$

where $H(u,v)$ is the bilinear form corresponding to the quadratic form $H$ defined above; viz., $H(u,v) = H(u+v) - Hu - Hv$. In this case we have $H(X_1,X_2) = -2B(Y_1,Y_2) + 2B(Z_1,Z_2)$.

We claim that $x \varepsilon \operatorname{ad}\mathfrak{g}$ implies ${}^{t}x \varepsilon \operatorname{ad}\mathfrak{g}$. First observe that for $Y \varepsilon \mathfrak{g} \cap \mathfrak{g}_K$ we have ${}^{t}\operatorname{ad}Y = -\operatorname{ad}Y$ and for $Z \varepsilon \mathfrak{g} \cap \sqrt{-1}\,\mathfrak{g}_K$ we have ${}^{t}(\operatorname{ad}z) = \operatorname{ad}z$. The proofs are straight forward. As the sum of two self-adjoint families

$\operatorname{ad}\mathfrak{g}$ is itself self-adjoint

Thus $\operatorname{ad}\mathfrak{g}$ is represented as a family of matrices closed under transpose and the adjoint group $\operatorname{Ad}\mathfrak{g}$ is stable under transpose. From the considerations above we see that $\operatorname{Ad}\mathfrak{g}$ can be written:

$$\operatorname{Ad}\mathfrak{g} = (\operatorname{Ad}\mathfrak{g} \cap O)\cdot(\operatorname{Ad}\mathfrak{g} \cap P).$$

# LECTURE XXXIII

We know that the adjoint group, $G$, of a semi-simple analytic group may be written topologically as $G = K \times E$ where $K$ is a compact subgroup and $E$ a euclidean space. We are now going to see how this result can be made to bear on an arbitrary semi-simple analytic group.

If $G$ is an arbitrary analytic group, then the adjoint group of $G$ is by definition $\mathrm{Ad}\, G = \{\mathrm{Ad}\, x;\ x \in G\}$-where $\mathrm{Ad}\, x$ is the differential at 1 (the identity in $G$) of the inner automorphism $\Gamma_x : g \longrightarrow x^{-1}gx$.

Proposition 1 The kernel of the adjoint representation Ad is the center of $G$.

Proof ,Identifying $\dot{G}$ with the tangent space to $G$ at 1, we argue as follows:

If $\mathrm{Ad}\, x(X) = X$ for all $X \in \dot{G}$ then

$$e^{\mathrm{Ad}\, x(X)} = x^{-1}e^{X}x = e^{X}$$

Thus $x$ commutes with all elements in a neighborhood of 1. Since $G$ is connected we have $xy = yx$ for all $y \in G$ and $x \in Z(G)$ = center of $G$.

If $x \in Z(G)$ we have $\mathrm{Ad}\, x$ = differential at 1 of the identity map, and hence $\mathrm{Ad}\, x =$ = the identity.

Thus $\mathrm{Ad}\, G = G/Z(G)$.

Corollary If $G$ is semi-simple then $Z(G)$ is discrete

Proof First, we observe that $Z(G)$ is closed and it is certainly normal and solvable (in fact-abelian).

The connected component of $\{1\}$ in $Z(G)$ is also closed and since $G$ admits no connected, normal, solvable subgroups other than $\{1\}$ we have
$\{1\}$ = connected component of the identity in $Z(G)$.
Since $Z(G)$ is a closed subgroup of an analytic group it is a Lie group and the connected component $\{1\}$ is thus openin $Z(G)$.

The canonical map $G \rightarrow G/Z(G) = Ad(G)$ is a covering map. We diverge for a few moments to look at a slightly more general situation, which we will have occasion to meet many times in the succeeding lectures. Let $G$ be an analytic group. Let $C$ be a closed subgroup and let $C_1$ be the connected component of 1 in $C$. The canonical map $\pi : G/C_1 \rightarrow G/C$ is a covering map. We have the commutative diagram

$$\begin{tikzcd}
G/C_1 \arrow[rr, "\pi"] \arrow[dr] & & G/C \\
& G/C_1/C/C_1 \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

where $\varphi$ is a homeomorphism. As $C/C_1$ is discrete we see that $\pi$ is a local homeomorphism.) The map $R_c : xC_1 \rightarrow xC_1c$ where $c \varepsilon C$ maps $G/C_1$ onto itself, and, as $C_1$ is normal in $C$, we may write $xC_1c = xcC_1$, and we see that $R_c$ commutes with $\pi$. We have $R_{c_1c_2} = R_{c_2} \circ R_{c_1}$ and the group $\{R_c; c \varepsilon C\}$, which is isomorphic to $C/C_1$ is the group of deck transformations of the covering.

Returning to the case at hand, let $G$ be a semi-simple analytic group. We have $\operatorname{Ad} G = K \times E$. Let $\hat{K}$ be the inverse image of $K$ with respect to the covering $\pi : G \to G/Z(G) = \operatorname{Ad}(G) = K \times E$. There is a homeomorphism $\alpha$ which makes the following diagram commutative:

$$\begin{tikzcd}
G \arrow[r, "\operatorname{Ad}"] \arrow[d, "\hat{\omega}"'] & \operatorname{Ad} G \arrow[d, "\omega"] \\
G/\hat{K} \arrow[r, "\alpha"] & \operatorname{Ad} G/K .
\end{tikzcd}$$

From the above discussion we have, where $\hat{K}_1$ = connected component of 1 in $\hat{K}$, that the group of deck transformations of $G/\hat{K}_1 \to G/\hat{K}$ is $\hat{K}/\hat{K}_1$. But since $G/\hat{K} \approx \operatorname{Ad} G/K \approx E$, which is simply-connected, $\hat{K} = \hat{K}_1$; i. e., $\hat{K}$ is connected.

We now show how we can choose $\hat{E}$ in $G$ such that $\operatorname{Ad} : \hat{E} \to E$ is a homeomorphism and $G = \hat{K} \times \hat{E}$ (topologically). Take $\hat{E}$ to be the connected component of 1 in $\operatorname{Ad}^{-1}(E)$. Since $\hat{E}$ is a covering space of the euclidean space $E$, it is a one-to-one covering; i. e. the covering map is a homeomorphism. Consider the map $\hat{K} \times \hat{E} \to G$ given by $(k,e) \to ke$. We claim that this map is one-to-one -- $k_1 e_1 = k_2 e_2$ implies $\operatorname{Ad} k_1 \operatorname{Ad} e_1 = = \operatorname{Ad} k_2 \operatorname{Ad} e_2$; therefore, $\operatorname{Ad} k_1 = \operatorname{Ad} k_2$ and $\operatorname{Ad} e_1 = \operatorname{Ad} e_2$. Since $\operatorname{Ad}$ is one-to-one on $\hat{E}$ we have $e_1 = e_2$ and then $k_1 = k_2$. The map can also be seen to be surjective by looking at the diagram

$$\begin{tikzcd}
\hat{E} \arrow[r] \arrow[d, "\text{onto}"'] & G \arrow[r] \arrow[d] & G/\hat{K} \arrow[d, "\wr"'] \\
E \arrow[r] \arrow[rr, bend right, "\text{onto}"'] & \mathrm{Ad}\ G \arrow[r] & \mathrm{Ad}\ G/K
\end{tikzcd}$$

to show that the map $\hat{E} \longrightarrow G/\hat{K}$ is onto. Thus, given any $g \in G$, there is an element $e \in \hat{E}$ such that the coset $\hat{K}g = \hat{K}e$, i. e., $g = ke$ with $k \in \hat{K}$. Thus $\hat{K} \times \hat{E}$ is homeomorphic by the above map to $G$ and we state the result:

> Theorem The semi-simple analytic group $G$ has the form $G = \hat{K} \times \hat{E}$ (topologically where $\hat{E}$ is homeomorphic to euclidean space and $\hat{K}$ is a covering group of a compact group.

We will study compact semi-simple groups and their coverings in detail - the above theorem suggests the importance of such information to the general theory of semi-simple groups.

# LECTURE XXXIV

We now study compact semi-simple analytic groups.

Definition Let $G$ be an analytic group. An element $x$ in $G$ is called regular if and only if $\operatorname{Ad} x$ has 1 as an eigenvalue with the minimum possible multiplicity. More explicitly; let $r$ be the largest positive integer so that $(\lambda-1)^r$ divides $\det(\operatorname{Ad} x-\lambda)$. For any element $x$, let $\mu(x)$ be the multiplicity of 1 as an eigenvalue in $\operatorname{Ad} x$. We call $x$ regular, then, if $\mu(x) = r$. We say $x$ is singular if $\mu(x) > r$

Remark $\det(\operatorname{Ad} x-\lambda) = \sum_{k=0}^{n} (\lambda-1)^{n-k} c_k(x)$. Thus, for any neighborhood $U$ of 1 in $G$ we have $\inf_{x\varepsilon U} \mu(x) = \inf_{x\varepsilon G} \mu(x)$. Then we can determine $r$ by looking only at a neighborhood of 1. For a suitable neighborhood $\mathcal{U}$ of 0 in $\dot{G}$ $\mu(\exp X)$ is the multiplicity of 0 as an eigenvalue in $\operatorname{ad} X$ as $\operatorname{Ad} \exp X = e^{\operatorname{ad} X}$. Thus $r = \operatorname{rank} \dot{G}$. Note it is not in general the case that $\exp X$ is regular in $G$ when $X$ is regular in $\dot{G}$.

Proposition 1 Let $T$ be an analytic subgroup of the compact semi-simple analytic group $G$. The Lie algebra $\dot{T}$ is a Cartan subalgebra of $\dot{G}$ if and only if $T$ is a maximal connected abelian subgroup of $G$.

Proof $(\longrightarrow)$ Assume $\dot{T}$ is a Cartan subalgebra. Then $\dot{T}$ is abelian and maximal nilpotent. Hence it is a

maximal abelian subalgebra. Thus $T$ is a maximal connected abelian analytic subgroup. As any abelian connected subgroup $A$ is contained in a maximal abelian connected subgroup $M$, and $M = \overline{M}$, whence $M$ is analytic, it follows that $T$ is a maximal connected abelian subgroup of $G$.

(←) Given that $T$ is a maximal connected abelian subgroup, we argue as above that it is a maximal connected abelian analytic subgroup. Thus $\dot{T}$ is a maximal abelian subalgebra of $\dot{G}$

The killing form of $\dot{G}$ is negative-definite and adjoint invariant; thus, if $V$ is a subspace of $\dot{G}$ stable under Ad $T$ then the orthogonal complement (with respect to the Killing form) $V^{\perp}$ is ad $T$ invariant also. In particular, let $\dot{T}^{\perp}$ be the orthogonol complement to $\dot{T}$ and let $\dot{N}$ be the nomalizer of $\dot{T}$. We have

$$[\dot{T}, \dot{T}^{\perp} \cap \dot{N}] \subset \dot{T} \cap \dot{T}^{\perp} = (0)$$

and $\dot{N} = \dot{T} + \dot{N} \cap \dot{T}^{\perp} = \dot{T}$. Thus $\dot{T}$ is maximal abelian and $\dot{T} = \dot{N}$ so $\dot{T}$ is a Cartan subalgebra.

Note $T$, being a closed subgroup of the compact group $G$, is itself compact and abelian and is thus a direct product of circle groups; i. e. $T$ is isomorphic (as Lie group) to a direct product of finitely many copies of $R/Z$ where $R$ = real numbers and $Z$ = the additive subgroup of integers. Such a group is called a toroid. The above proposition may be read to assert that the Cartan subalgebras in $\dot{G}$ are the

Lie algebras of maximal tori of $G$.

*Definition* Let $G$ be a compact analytic group. Let $T$ be a maximal toroid in $G$. We define the *diagram* $S(G)$ to be the set of singular elements in $T$. We will see later that this definition is independent of the choice of $T$.

Let $\dot{T}$ be the Lie algebra of the maximal toroid $T$. Since $\exp(a+b) = (\exp a)(\exp b)$ when $a$ and $b$ commute, exp is a homomorphism of $\dot{T}$ onto $T$— as it is, in general, a local homeomorphism we can identify the pair $(\dot{T}, \exp)$ with the simply-connected covering group of $T$. Specification of $S(G)$ is equivalent to the specification of $\exp^{-1}S(G) \cap \dot{T}$ and of the set $(\exp^{-1}1) \cap \dot{T}$. We also refer to this latter information as the *diagram of $G$* denoting $\dot{T} \cap \exp^{-1}S(G)$ by $D(G)$.

*Lemma* Let $G$ be a compact semi-simple analytic group and let $\mathfrak{G}$ denote the complexification of $\dot{G}$. Let $\mathfrak{H}$ denote complexification of the Cartan subalgebra $\dot{T}$, where $T$ is a fixed maximal toroid in $G$. For any root $\alpha$ on $\mathfrak{H}$ let $P_{\alpha,n}$ denote the subset of elements $h$ of $\dot{T}$ such that $\alpha(h) = 2\pi\sqrt{-1}\,n$.

Then $D(G)$ is the union of all the $P_{\alpha,n}$ for all roots $\alpha$ and all integers $n$.

*Proof* Note first that any root $\alpha$ takes on purely imaginary values on $\dot{T}$, since ad $h$ is skew-symmetric with respect to the killing form. The abelian family

of semi-simple endomorphisms $\{\mathrm{ad}\ H\}_{G/H}$ can be simultanuously diagonalized and

$$\mathrm{Ad}\ \exp h = \mathrm{diag}(\dots, e^{\alpha(h)}, \dots)$$

$\alpha$ ranging over all roots. The element $\exp h$ of $G$ is singular if and only if $\alpha(h) = 2\pi i n$ for some $\alpha$ and some $n$; i. e., $h \in D(G)$ if and only if $\alpha(h) = 2\pi i n$ for some $\alpha$ and some $n$.

Remark For $x \in G$ denote the centralizer of $x$ by $G^x$. The codimension of $G^x$ in $G$ is always even for we have $\dim G^x = \dim \dot{G}^x$ = the number of fixed points of $\mathrm{Ad}\ x$ = dimension of the eigenspace of $\mathrm{Ad}\ x$ corresponding to the eigenvalue 1. Now $\mathrm{Ad}\ x$ is orthogonal with respect to the Killing form which is negative definite and $\det \mathrm{Ad}\ x = 1$-hence $(-1)$ cannot occur an odd number of times as an eigenvalue ($\mathrm{Ad}\ x$ in triangular form is

$$\mathrm{Ad}\ X = \begin{pmatrix}
1 & & & & & & & & & & & & & \\
& 1 & & & & & & & & & & & & \\
& & 1 & & & & & & & & & & & \\
& & & \ddots & & & & * & & & & & & \\
& & & & -1 & & & & & & & & & \\
& & & & & -1 & & & & & & & & \\
& & & & & & -1 & & & & & & & \\
& & & & & & & \ddots & & & & & & \\
& & 0 & & & & & & -1 & & & & & \\
& & & & & & & & & a & & & & \\
& & & & & & & & & & \bar{a} & & & \\
& & & & & & & & & & & b & & \\
& & & & & & & & & & & & \bar{b} & \\
& & & & & & & & & & & & & \ddots
\end{pmatrix}$$

and so $-1$ occurs an even number of times). As the other eigenvalues occur in pairs (complex-conjugates) we see $\dim G - \dim \dot{G}^x$ is even.

In particllar, if $x$ is singular $\dim G^x \geq \operatorname{rank} G + 2$, since $\operatorname{rank} G = \inf_{x \varepsilon G} \dim G^x$.

Remark We shall call $P_{\alpha,n}$ "a singular plane" in $\dot{T}$.

$D(G)$ is thus the union of the singular planes in $\dot{T}$.

We continue our discussion of compact semi-simple analytic groups. In Lemma 1 and Theorem 1 below, $G$ is assumed to have these properties. Thereafter, it becomes convenient to assume that $G$ is the covering group of such a group. In the end it will turn out that $\pi(G)$ is finite and that $G$ is compact after all.

Lemma 1: Let $S$ be the set of singular elements in $T$ and let $G(S)$ be the orbit of $S$ under $T$. Then $G - G(S)$ is connected and $\pi_1(G) = \pi_1(G - G(S))$.

Proof: $S$ is a finite union of submanifolds in $T$ of $\text{codim} = 1$. For each $x \varepsilon S$, the orbit $Gx$ has a dimension equal to $\dim G/G^x \leq G/T - 2$. Therefore

$$\begin{aligned} \dim G(S) &\leq \dim S + \dim G/T - 2 \\ &\leq \dim T - 1 + \dim G/T - 2 \\ &\leq \dim G - 3 \end{aligned}$$

Since $\text{codim} G(S) > 1$ we have $G - G(S)$ connected. Because $G - G(S)$ is a finite union of submanifolds each of codimension at least 3, we can conclude $\pi_1(G) = \pi_1(G - G(S))$.

Note: The last statement can be proved by induction on the number of submanifolds removed from $G$ in forming $G - G(S)$.

Theorem 1: Every element in $G$ is conjugate to an element of $T$.

Proof: Consider the map $\psi : G \times (T - S) \longrightarrow G - G(S)$ given by $\psi(g,x) = gxg^{-1}$ . By our previous computations for complex semi-simple Lie algebras we can say that the rank of $\psi$ is the dimension of $G$ . It follows that the image of $\psi$ is open and it is also closed because $(T-S)$ is closed in $G - G(S)$ and $G$ is compact. Hence $\psi$ is onto.

The orbit of $T$ under $G$ contains $G - G(S)$ and is closed in $G$ . As the closure of $G - G(S)$ is $G$ we see that the orbit of $T$ under $G$ is $G$ itself.

The results of the previous discussion of compact semi-simple analytic groups may be formulated and proved for a covering group $G$ of such a group. We assume hereafter that $G$ is an analytic group which covers a compact semi-simple analytic group. For example, we have:

Theorem 1': Let the analytic group $G$ be a covering group of the compact semi-simple analytic group $G'$ . Let $T$ be a covering of a toroid $T'$ in $G'$ . Then every element of $G$ is conjugate to an element in $T$ .

Since $S(G)$ covers $S(G')$ , we have $\dim G(S) \leq \dim G - 3$ and $G - G(S)$ .

Proof: Consider the map

$$\varphi : G \times (T - S) \longrightarrow G - G(S)$$

defined by $\varphi(g,h) = \exp \operatorname{Ad} g(h)$ . The image of $\varphi$ is open since rank $\varphi = \dim G$ .

To show that the image of $\varphi$ is also closed we argue as follows:

$(\dot{T}-S)$ goes into the complement of the singular elements and the map $\varphi$ factors through $\operatorname{Ad} G \times (\dot{T}-S)$ as indicated by the diagram:

$$\begin{tikzcd}
G \times (\dot{T}-S) \arrow[r, "\varphi"] \arrow[d] & G - G(S) \\
\operatorname{Ad} G \times (\dot{T}-S) \arrow[ur, "\bar{\varphi}"] &
\end{tikzcd}$$

where $\bar{\varphi}: \operatorname{Ad} G \times (\dot{T}-S) \longrightarrow G - G(S)$ is defined by $\bar{\varphi}(g,h) = \exp g(h)$ . The image of $\varphi$ is the image of $\bar{\varphi}$ which is closed since $\operatorname{Ad} G$ is compact and $T - S$ is closed in $G - G(S)$ .

The rest of the argument is just like that of Theorem 1.

Corollary 1: $T$ contains $Z(G)$ .

Corollary 2: $G/T$ is simply connected and compact

Proof: Let $(\tilde{G},\pi)$ be the simply-connected covering group Let $\tilde{T}$ be the analytic subgroup such that $\pi(\tilde{T}) = T$ . Then

$$G/T = \tilde{G}/_{\pi^{-1}(T)} = \tilde{G}/_{\tilde{T}\cdot\pi^{-1}(1)} = \tilde{G}/\tilde{T}$$

and $\tilde{G}/\tilde{T}$ , being a simply-connected group modulo a connected subgroup, is simply connected. By Corollary 1, $G/T$ is a continuous image if $G/Z(G) = \operatorname{Ad} G$ , and thus $G/T$ is compact.

Corollary 3: If $T_1$ and $T_2$ are analytic subgroups with $\mathrm{Ad}\ T_1$ and $\mathrm{Ad}\ T_2$ maximal toroids (as usual we let Ad denote the adjoint representation $\mathrm{Ad}:G \longrightarrow \mathrm{Ad}(G)$ , then $T_1$ and $T_2$ are conjugate under an inner automorphism of $G$ .

Proof: Let $T$ be any analytic subgroup of $G$ with $\mathrm{Ad}\ T$ a maximal toroid in $\mathrm{Ad}\ G$ . Let $x$ be a regular element in $T$ . Then $T$ is the connected component of the identity in $G^x$ . The result now follows from the theorem.

Corollary 4: The diagram $D(G)$ is independent of the choice of $T$ .

Proof: An inner automorphism does not change the eigenvalues of an element and carries singular planes into singular planes.

Corollary 5: $G(S)$ is the set of singular elements in $G$ . $G(T-S)$ is the set of regular elements in $G$ .

Notation: We write $G_r$ for $G(T-S) = G - G(S)$ = the set of regular elements in the analytic group $G$ (where $\mathrm{Ad}\ G$ is compact).

Corollary 6: The maximal toroids in a compact Lie group $G$ are conjugate under an inner automorphism.

Proof: Clearly it suffices to assume $G$ connected. Since $Z_1$ , the connected component of the identity in the center of $G$ , lies in every maximal toroid, we may pass to the quotient group $G/Z_1$ , which is a compact analytic semi-simple group. Corollary 6 now follows from Corollary 3.

Note: The foregoing proof uses the fact that the radical of a compact analytic group is central. This in turn is a special case of the more general fact:

Exercise: A normal toroid in an analytic group is central. (Hint: Lift the inner automorphisms of the toroid to its simply connected covering group and observe that the elements of the kernel are left fixed.)

We introduce the group $\widetilde{N}$ , which will relate the fundamental group of $G_r$ to the diagram of $G$ in an explicit way.

Definition: Let $A$ and $B$ be sets and let $N$ be a group operating in $A$ on the on the right and in $B$ on the left. Then $A \times_N B$ is the set of equivalence classes of $A \times B$ taken with respect to the equivalence relation

$$(an,b) \equiv (a,nb)$$

that is, $A \times_N B$ is the set of orbits $\{(an^{-1},nb) ;\ n \in N\}$ . If $A$ and $B$ are topological then $A \times_N B$ has a canonical topological structure which is separated if the above $N$-orbits are closed. If moreover $A$ is a principal $N$-bundle, then $A \times_N B$ is a bundle over $A/N$ with fiber $B$ .

Let $T$ be an analytic subgroup of $G$ such that $\operatorname{Ad} T$ is a maximal toroid in $\operatorname{Ad} G$ . Let $N(T)$ denote the normalizer of $T$ in $G$ .

Lemma 1: For any $g \in N(T)$ , let $\theta$ denote the map assigning to $g$ the restriction to $T$ of the inner automorphism $T_g : x \longrightarrow g x g^{-1}$ . Then $\theta$ is a homomorphism with kernel $T$ .

Proof: That $\theta$ is a homomorphism is clear. It remains to prove that $\operatorname{Ker} \theta = T$ . In as much as $\operatorname{Ker} \theta \supset T \supset Z(G)$ , no generality is lost in dividing out $Z(G)$ ; thus we may assume hereafter that $G$ and $T$ are compact. Our problem is to prove that $T$ is a maximal abelian

subgroup of $G$.

Suppose $b$ is an element with $bt = tb$ for all $t \in T$. Then $b$ is in the center of $Z(b)$, the centralizer of $b$, and $T \subset Z(b)$. By Theorem 1 of XXXV, $b$ lies in the connected component of the identity in $Z(b)$, and hence by *ibid*. Corollary 1, in every maximal toroid of $Z(b)$. In as much as $T$ is a maximal toroid in $Z(b)$, we have $b \in T$. Thus $\operatorname{Ker} \theta = T$.

*Lemma 2*: The adjoint representation maps $N(T)$ into the Weyl group of the complexification of $\dot{G}$ and induces an isomorphism $N(T)/T \cong W(\dot{G} \otimes \mathbb{C})$.

*Proof*: $W(\dot{G} \otimes \mathbb{C})$ operates transitively on the Weyl chambers of $\sqrt{-1}\,\dot{T}$ (the root vectors of $\dot{T}$ are pure imaginary). Let $C$ be a fixed chamber. $C$ is defined by $\alpha_1 > 0, \ldots, \alpha_r > 0$ where $\alpha_1, \ldots, \alpha_r$ is a fundamental system of roots for $\dot{G}$. Given $n \in N(T)$, we want to show that $\operatorname{Ad} n$ restricted to $\sqrt{-1}\,\dot{T}$, denoted $\operatorname{Ad} n\,|\sqrt{-1}\,\dot{T}$, is in $W(\dot{G} \otimes \mathbb{C})$. No generality is lost in assuming $\operatorname{Ad} n(C) = C$ and $\operatorname{Ad} n(h_1) = h_1$ where $\alpha_i(h_1) = 1$ $(i = 1,2,\ldots,n)$.

Take $x_0 = n \cdot \exp c\, h_1$. $\operatorname{Ad} n$ permutes $\alpha_1, \alpha_2, \ldots, \alpha_r$ and permutes the roots. Writing the latter permutation as a product of disjoint cycles, consider the cycle

$$\beta_1 \xrightarrow{\operatorname{Ad} n} \beta_2 \xrightarrow{\operatorname{Ad} n} \beta_3 \xrightarrow{\operatorname{Ad} n} \cdots \xrightarrow{\operatorname{Ad} n} \beta_k \xrightarrow{\operatorname{Ad} n} \beta_1 .$$

If $k \neq 1$, then $\mathrm{Ad}(n \cdot \exp ch_1)$ has no eigenvalue equal to 1 on the span $\mathbb{R}e_{\beta_1} + \ldots + \mathbb{R}e_{\beta_k}$. If, however, we have $\mathrm{Ad}\, n(\beta) = \beta$, then

$$\mathrm{Ad}(n \cdot \exp ch_1)e_\beta = e^{c\beta(h_1)}\, \mathrm{Ad}\, n(e_\beta)$$

and $\beta(h_1) \neq 0$. So for a general $c$, $\mathrm{Ad}\,(n \exp ch_1)$ has no eigenvalue equal to 1 on $\sum_{\text{all roots } \beta} \mathbb{R}e_\beta$. $\mathrm{Ad}\, n$ has eigenvalue 1 with multiplicity $r$ on $\sqrt{-1}\,\dot{T}$. It follows that $\mathrm{Ad}\, n(h) = h$ for all $h \in \sqrt{-1}\,\dot{T}$ and $\mathrm{Ad}\, n|T$ is contained in $W(\dot{G} \otimes \mathbb{C})$. That the mapping of $N(T)$ into $W(\dot{G} \otimes \mathbb{C})$ is surjective follows from the explicit form of the inner automorphisms inducing the Weyl reflections; e.g., they can be taken from any compact real form of $\dot{G} \otimes \mathbb{C}$ (cf. Remark following Theorem 2 of Lecture XXX and Proposition, Lecture XXIX).

Exercise: The group of outer automorphisms $\mathrm{Aut}\,\mathcal{G}/\mathrm{Ad}\,\mathcal{G}$ is isomorphic to the group of automorphisms of the Dynkin diagram

Set $N = \theta(N(T))$. Let $\tilde{N}$ denote the group of all transformations of $\dot{T}$ which lies over the elements of $N$. That is, $\tilde{\varphi} \in \tilde{N}$ if and only if the following diagram is commutative.

$$\begin{tikzcd}
\dot{T} \arrow[r, "\tilde{\varphi}"] \arrow[d, "\exp"'] & \dot{T} \arrow[d, "\exp"] \\
T \arrow[r, "\varphi"'] & T
\end{tikzcd}$$

for some $\varphi \in N$. Let $\rho$ denote the homomorphism $\tilde{\varphi} \longrightarrow \varphi$ of $\tilde{N}$ onto $N$.

Let $N(T)$ operate on $G/T$ by right translation and on $T$ via $\theta$. The induced operations of $N$ on $G/T$ and $T-S$ allow us to form $G/T \times_{N} T-S$ and also $G/T \times_{\tilde{N}} \dot{T}-D(G)$ in the natural way. It should be observed that $G/T$ is a principal $N$ bundle.

Proposition: $G_r = G/T \times_{N} (T-S) = G/T \times_{N} (\dot{T}-D(G))$

Proof: Consider the map $\varphi: G \times (T-S) \longrightarrow G_r$ defined above; i.e.,

$$\varphi(g,t) = gtg^{-1} .$$

For any $n \in N(T)$ we have

$$\varphi(gn,t) = gntn^{-1}g^{-1} = \varphi(g,ntn^{-1}) .$$

Hence $\varphi$ factors --

$$\begin{tikzcd}
G \times (T-S) \arrow[rr, "\varphi"] \arrow[d] & & G_r \\
G/T \times_{N} (T-S) \arrow[urr, "\bar{\varphi}"'] & &
\end{tikzcd}$$

with $\bar{\varphi}$ surjective (by Corollary 5 of XXXV). $\bar{\varphi}$ is also injective since

$$g_1t_1g_1^{-1} = g_2t_2g_2^{-1} \quad \text{implies}$$

$$(g_2^{-1}g_1)t_1(g_2^{-1}g_1)^{-1} = t_2 ,$$

whence $g_2^{-1}g_1 \in N(T)$ and $(g_1,t_1) = (g_2(g_2^{-1}g_1),t_1) \sim (g_2,t_2)$. Since $\bar{\varphi}$ is a one-to-one continuous

mapping of separable locally compact spaces it is a homeomorphism. We have the first part of the equation.

The proof of the second part is similar. We consider tha map $\psi : G \times \dot{T} - D(G) \longrightarrow G/T \underset{N}{\times} T-S$ defined by

$$\psi(g,h) = (gT, \exp h) \mod N$$

i.e., $\psi(g,h)$ is the element of $G/T \underset{N}{\times} T-S$ which contains $(gT, \exp h)$. For any $n \in N(T)$ and for any $\tilde{n} \in \tilde{N}$ such that $\rho(\tilde{n}) = \theta(n)$, we see

$$\psi(gn,h) = \psi(g, \operatorname{Ad} n(h)) = \psi(g, \tilde{n}(h)) .$$

Thus $\psi$ induces a map $\bar{\psi} : G/T \underset{\tilde{N}}{\times} (\dot{T}-D(G)) \longrightarrow G/T \underset{N}{\times} (T-S)$. Given $n_1, n_2 \in N(T)$ and $\tilde{n}_1, \tilde{n}_2 \in \tilde{N}$ with $\rho(\tilde{n}_1) = \theta(n_1)$, $\rho(\tilde{n}_2) = \theta(n_2)$, we conclude from $\psi(g, \tilde{n}_1 h) = {}$ $= \psi(g, \tilde{n}_2 h)$ that $\psi(gn_1, h) = \psi(gn_2, h)$, that $\theta(n_1)$ $= \theta(n_2)$, and hence $\rho(\tilde{n}_1) = \rho(\tilde{n}_2)$. Thus $\bar{\psi}$ is injective. It is now clear that $\bar{\psi}$ is a homeomorphism.

# LECTURE XXXVII

We are now ready to discuss the fundamental group of the semi-simple analytic group $G$ whose adjoint group $\mathrm{Ad}\, G$ is compact (this is equivalent to saying that $G$ is an analytic group which covers a compact semi-simple analytic group $G'$). First we clarify our notation.

$T$ is an analytic subgroup of $G$ such that $\mathrm{Ad}\, T$ is a maximal toroid and $\dot{T}$ is its Lie algebra in $\dot{G}$. The set of singular elements in $T$ is denoted by $S$ and $D(G) = \exp^{-1}(S) \cap \dot{T}$. The singular plane $P_{\alpha,n} = \{h;\, h \in \dot{T} \text{ and } \frac{\alpha(h)}{2\pi i} = h\}$ contains an element $h_{\alpha,n}$ perpendicular to $P_{\alpha,0}$ (with respect to the Euclidean structure defined by the Killing form). We define a translation $\gamma_{\alpha,n}$ of $\dot{T}$; viz., $\gamma_{\alpha,n}(h) = h + h_{\alpha,n}$. We denote by $r_{\alpha,n}$ the reflection in the plane $P_{\alpha,n}$. Denote by $W$ the *Weyl group* of the group $G$; i. e., the group of transformations of $\dot{T}$ generated by the $r_{\alpha,n}$ (for all roots $\alpha$ and all integers $n$). The Weyl group $W$, while it depends on $G$ is the same for all groups with Lie algebra $\dot{G}$ since the singular plane $P_{\alpha,n}$ is determined by $\mathrm{Ad}\, G$. Let $W_0$ be the subgroup of $W$ keeping $0$ fixed; i. e., $W_0$ is the subgroup generated by $r_{\alpha,0}$ (all roots $\alpha$) and can be identified with the Weyl group of the Lie algebra $\dot{G} \otimes \mathbb{C}$. $N(T)$ is as before the normalizer of $T$ in $G$ and $N$ is the group which consists of the restrictions to $T$ of inner-automorphisms by elements of $N$. We denote by $\tilde{N}$, the set of all continuous maps of $\dot{T}$ which cover elements of $N$;

i. e., $\tilde{\varphi} \in \tilde{N}$ if and only if there is a $\varphi \in N$ with the following diagram commutative:

$$\begin{tikzcd}
\dot{T} \arrow[r, "\tilde{\varphi}"] \arrow[d, "\exp"'] & \dot{T} \arrow[d, "\exp"] \\
T \arrow[r, "\varphi"] & T
\end{tikzcd}$$

such a $\varphi$ is unique. Denote the homomorphism $\tilde{\varphi} \longrightarrow \varphi$ by $\rho : \tilde{N} \longrightarrow N$. we can reformulate Lemma 2 of Lecture XXXVI as follows:

Lemma 1 N is isomorphic to $W_0$ under the map

$$\delta : \varphi \longrightarrow d\varphi_1$$

where $d\varphi_1$ denotes the differential of $\varphi$ at the identity element.

Since $\exp \operatorname{Ad} n(h) = n(\exp h)n^{-1}$ for $n \in N$ and $h \in \dot{T}$ we have

Lemma 2 $\rho \circ \delta$ is the identity on N.

We conclude the

Corollary $\tilde{N} = W_0 \cdot \operatorname{Ker} o$ (semi-direct)

Proof Set $\tilde{\Gamma} = \operatorname{Ker} \rho$. Lemma 2 shows that every element of $\tilde{N}$ may be written as the product of an element in $\delta(N) = W_0$ with an element in $\tilde{\Gamma}$

Now $\gamma \in \tilde{\Gamma}$ if and only if $\gamma(h) = h + he$ where $\exp(he) = 1$ = identity of T. Hence $W_0 \cap \tilde{\Gamma}$ = the identity and we have

$$\tilde{N} = W_0 \cdot \tilde{\Gamma} \quad \text{(semi-direct)}$$

We remark that $\tilde{\Gamma}$ is normal in $\tilde{N}$ (being the kernel of a homomorphism. Also, the elements of $\tilde{N}$ are rigid motions as the elements of $W_0$ preserve the Killing form and the elements of $\tilde{\Gamma}$ are translations.

We define the group $\Gamma$ to be the group of transformations of $\dot{T}$ generated by the translations $\{\gamma_{\alpha,2n};\ \alpha$ a root and $n$ an integer$\}$.

Lemma 3

$$W = W_0 \cdot \Gamma \quad \text{(semi-direct)}$$

Proof We certainly have $W_0 \subset W$. Now

$$r_{\alpha,n} = \gamma_{\alpha,n} \circ r_{\alpha,0} \circ \gamma^{-1}_{\alpha,n}$$

$$= r_{\alpha,0}(r^{-1}_{\alpha,0}\ \gamma_{\alpha,n} r_{\alpha,0}) \circ \gamma^{-1}_{\alpha,n}$$

$$= r_{\alpha,0}(\gamma_{-\alpha,n}) \circ \gamma_{-\alpha,n}$$

$$= r_{\alpha,0}\ \gamma_{\alpha,-2n},$$

and from this we conclude that $\gamma_{\alpha,-2n} \in W$, whence $\Gamma \subset W$, and $r_{\alpha,n} \subset W_0 \cdot \Gamma$. Since $\Gamma$ is a normal subgroup of $W$ the set of products $W_0 \cdot \Gamma$ is a subgroup. As $W_0 \cdot \Gamma$ contains a set of generators for $W$ we see $W_0 \cdot \Gamma = W$. Furthermore $W_0 \cap \Gamma = (1)$ and so $W = W_0 \cdot \Gamma$ (semi-direct).

Lemma 4. $\Gamma \subset \tilde{\Gamma}$ and $W \subset \tilde{N}$

Proof: The second assertion follows from the first. For that part we write:

$$r_{\alpha,0} = \mathrm{Ad}_{\dot{T}}\, n$$

with $n \in N$ and

$$\exp h_{\alpha,1} \cdot n \exp -h_{\alpha,1} = n.$$

Then

$$n(\exp h_{\alpha,1})n^{-1} = \exp h_{\alpha,1} \cdot$$

$$\exp [r_{\alpha,0}(h_{\alpha,1})] = \exp h_{\alpha,1} \ .$$

We find that

$$\exp (-h_{\alpha,1}) = \exp(h_{\alpha,1})$$

and so $\exp (2h_{\alpha,1})$ is the identity. It follows that $2\gamma_{\alpha,1} \in \tilde{\Gamma}$ and hence $\gamma_{\alpha,2n}$ is contained in $\tilde{\Gamma}$.

*Lemma 5.* The coset space $\tilde{N}/W$ is of finite order.

*Proof* The elements of $\tilde{N}/W = W_0 \cdot \tilde{\Gamma}/W_0 \cdot \Gamma$ are in one-to-one correspondence with these of the set $\tilde{\Gamma}/\Gamma$. The set $\tilde{\Gamma}/\Gamma$ is finite as both $\tilde{\Gamma}$ and $\Gamma$ are discrete subgroups of $\dot{T}$ each containing a basis for $\dot{T}$.

*Proposition* Let $F_r$ be a connected component in $\dot{T} - D(G)$. Then

a) $F_r$ ia a fundamental domain for the action of $W$ in $\dot{T} - D(G)$; i. e., the mapping $\psi : W \times F_r \longrightarrow \dot{T} - D(G)$ given by

$$\psi(w,h) = w(h)$$

is a homeomorphism

b) If $G$ is simply-connected $W = \tilde{N}$

c) In general $\pi_1(G) = \tilde{N}/W$

A proof of statement a) can be obtained from the general theory of groups generated by reflections. We prove a) here by a special argument. In fact we prove a) and b) simultaneously.

Consider the following sequence of (obvious) maps:

$$G/T \times F_r \longrightarrow G/T \times (\dot{T} - D(G)) \longrightarrow G/T \times_W (\dot{T} - d(G)) \longrightarrow G/T \times_{\tilde{N}} (\dot{T} - D(G))$$

composing the first two maps, we get the sequence of covering spaces

$$(*) \quad G/T \times F_r \longrightarrow G/T \times_W (\dot{T} - D(G)) \longrightarrow G/T \times_{\tilde{N}} (\dot{T} - D(G)).$$

In case $G$ is simply connected, then so is $G_r =$ $= G/T \times_{\tilde{N}} (\dot{T} - D(G))$ (XXXV - Lemma 1) and hence

$G/T \times F_r = G/T \times_W (\dot{T} - D(G)) = G/T \times_{\tilde{N}} (\dot{T} - D(G))$. Thus $\tilde{N} = W$ if $G$ is simply connected. Inasmuch as $G/T$ and the Weyl group of $G$ are the same for all covering groups of $G$, we see that for any analytic semi-simple group $G$ with Ad $G$ compact $G/T \times_W (\dot{T} - D(G))$ is simply connected.

It is clear that $W$ permutes transitively on the connected components in $\dot{T} - D(G)$. Thus $\dot{T} - D(G) = WF_r$ and from

$$G/T \times F_r = G/T \times_W WF_r$$

we conclude that $F_r$ ia a fundamental domain for $W$.

c) We have $\pi_1(G) = \pi_1(G - G(S)) = \pi_1(G_r) = \pi_1(G/T \times_{\tilde{N}} (\dot{T} - D(G))$.

As $G/T \times_W \dot{T} - D(G)$ is simply connected we have $\pi_1(G/T \times_{\tilde{N}} (\dot{T} - D(G)) = \tilde{N}/W$, the covering group of $G/T \times_W (\dot{T} - D(G)) \longrightarrow G/T \times_{\tilde{N}} (\dot{T} - D(G))$ in (*). We note that $W$ being generated by reflections is a normal subgroup of $\tilde{N}$.

We have the following result on the order of $\tilde{N}/W = \pi_1(G)$.

Theorem Let $F_r$ be a connected component in $\dot{T} - D(G)$. Then the number of elements in $\pi_1(G)$ is the number of vertices of $F_r$ which lie in $\exp^{-1}(1)$ (i. e., in the kernel of $\exp: \dot{T} \longrightarrow T$).

Proof Since the $F_r$'s are permuted transitively by $W$, there is no loss of generality in assuming $F_r$ has a vertex at 0. Let $N^*$ be the subgroup of $\tilde{N}$ keeping $F_r$ invariant. Then $\tilde{N} = N^* W$ and $N^* \cap W =$ (the identity) since $F_r$ is a fundamental domain for $W$.

Consider the map $f: N^* \longrightarrow \dot{T}$ defined by $f(n) = n(0)$. The map $f$ is injective, for if $f(n_1) = f(n_2)$ then

$$\begin{aligned} f(n_1^{-1}n_2) &= n_1^{-1}n_2(0) \\ &= n_1^{-1}n_1(0) \\ &= 0 . \end{aligned}$$

Now $n_1^{-1}n_2$ is the lift of an element of $N$ which takes 1 onto 1. Since $\rho(W_0) = N$ it follows that $n_1^{-1}n_2$ differs from an element in $W_0$ by an element of $\tilde{\Gamma} = \ker \rho$

(which consists of translations of $\dot{T}$). As $n_1^{-1}n_2(0) = 0$ the translation must be the identity and $n_1^{-1}n_2 \in W_0$. Hence $n_1^{-1}n_2 \in N^* \cap W_0 =$ (the identity) and $n_1 = n_2$.

Now $f(N^*)$ consists of vertices of $F_r$ and $f(N^*) \subset \exp^{-1}(1) \cap \bar{F}_r$. Given any $h \in \dot{T}e \cap \bar{F}_r$ we know that the translation by $h$, $\gamma_h$, belongs to $\tilde{N}$. Now $\gamma_h^{-1} F_r$ has $0$ as a vertex. Therefore, we can find $w \in W_0$ such that $w\gamma_h^{-1}F_r = F_r$. It follows that $w\gamma_h^{-1}$ and $\gamma_h w^{-1}$ are contained in $N^*$ and so $\gamma_h w^{-1}(0) = \gamma_h(0) = h$ and we have proved that $f(N^*) = \exp^{-1}(1) \cap \bar{F}_r$.

It follows that the number of elements in $N^*$ is the number of elements in $\exp^{-1}(1) \cap \bar{F}_r$ which is the number of vertices of $F_r$ belonging to $\exp^{-1}(1)$. Since $\tilde{N} = N^*W$ (semi-direct) the number of elements in $N^*$ is the order of $\tilde{N}/W = \pi_1(G)$.

Remark The region $F_r$ is a simplex with $\ell + 1$ faces (where $\ell =$ rank G). If for example, we take for $F_r$ the region bounded on $\ell$ sides by the planes $\{P_{\alpha_i,0}\}_{i = 1,2,\dots,\ell}$ for $\alpha_1,\alpha_2,\dots,\alpha_\ell$ a simple system of roots, then the last side is on the plane $P_{\bar{\beta},1}$ where $\bar{\beta}$ is the highest root.

The reason is that the highest root $\bar{\beta} = \Sigma_i \bar{m}_i\alpha_i$ has the property: For any root $\beta = \sum_i m_i\alpha_i$ we have $\bar{m}_1 \geq m_1$, $\bar{m}_2 \geq m_2, \dots, \bar{m}_\ell \geq m_\ell$

Proof. Let $\mathcal{G}$ denote the complexification of the Lie algebra of G. and let $\mathcal{E}$ denote the associative subalgebra of $\mathrm{Hom}(\mathcal{G},\mathcal{G})$ that is generated by ad $\mathcal{G}$.

Let $\mathcal{E}_+$ (resp. $\mathcal{E}_-$) denote the subalgebras of $\mathcal{E}$ generated by $\dot{T}$ and ad $x_\beta$ with $\beta > 0$ (resp. $\beta < 0$). Clearly $\mathcal{E} = \mathcal{E}_- \mathcal{E}_+$, where $\mathcal{E}_- \mathcal{E}_+$ consists of sums of terms of the form $g \cdot p$ with $g \in \mathcal{E}_-$ and $p \in \mathcal{E}_+$.

Without lost of generality, we may assume that $\mathcal{G}$ is simple. Then $\mathcal{G} = \mathcal{E} x_{\beta^0} = \mathcal{E}_- x_{\beta^0}$ is spanned by root vectors of the form $x_\beta$ with $\beta = \beta^0 - n_1\alpha_1 \ldots - n_\ell \alpha_\ell$, with $n_\ell \geq 0$.

Exercises

1. A complex analytic semi-simple group has a finite covering group
   (Hint: consider a compact real form)

2. Any representation of a semi-simple Lie algebra $\mathcal{G}$ over a field K of characteristic zero is fully reducible
   (Hint: can assume $K \subset \mathbb{C}$. Consider the associate representation of a compact real form of $\mathcal{G} \underset{K}{\otimes} \mathbb{C}$).

We now take up the study of the topology of $G/C$ where $C$ is a closed subgroup of the analytic group $G$. We will follow the account in Memoirs of the American Mathematical Association, Number 14, (1955) by G.D. Mostow.

Let $\mathcal{S}$ be the linear space of real $n \times n$ symmetric matrices. Denote by $P$ the space of real $n \times n$ symmetric positive definite matrices. On $\mathcal{S}$ we introduce the bilinear form $\operatorname{tr} XY$. The eigenvalues of an element of $\mathcal{S}$ being real we have $(\operatorname{tr} XX) \geq 0$ and $\operatorname{tr} XY$ is a symmetirc positive semi-definite bilinear form on $\mathcal{S}$.

Let $\mathcal{M}$ be the space of all $n \times n$ real matrices. For $g$ and $X$ in $\mathcal{M}$ we define maps $R_X$, $L_X$, and $D_X$ of $\mathcal{M}$ into itself by:

$$R_X : g \longrightarrow gX$$
$$L_X : g \longrightarrow Xg$$
$$D_X = L_X - R_X$$

For $X \in \mathcal{S}$ we define the map $\gamma_X : \mathcal{S} \longrightarrow \mathcal{S}$ by:

$$\gamma_X(Y) = \frac{d}{dt}\Big|_{t=0} \exp -X/2 \exp(X+tY) \exp -X/2 .$$

Lemma 1: $\gamma_X = \sinh(D_{X/2})/D_{X/2}$ and $\gamma_X$ is self-adjoint on $\mathcal{S}$.

Proof: Every matrix $X$ commutes with its exponential; i.e.,

$$X\cdot(\exp X) = (\exp X)\cdot X \quad .$$

Let $X(t)$ be a differentiable path in $\mathcal{S}$. Then

$$X(t)\cdot(\exp X(t)) = (\exp X(t))\cdot X(t) \quad ,$$

whence

$$\dot{X}\cdot\exp X + X(\dot{\exp}\, X) = (\dot{\exp}\, X)\cdot X + (\exp X)\cdot\dot{X}$$

i.e.,

$$X\cdot\dot{\exp}\,X - (\dot{\exp}\,X)\cdot X = (\exp X)\cdot\dot{X} - \dot{X}\exp X \quad .$$

Multiplying both sides of the last equation by $\exp(-X/2)$ on both the left and the right and using $(\exp(-X/2))\cdot X = X\cdot\exp(-X/2)$ we get

$$X\exp(-x/2)\dot{\exp}X\exp(-X/2) - \exp(-X/2)\dot{\exp}X\exp(-X/2)X =$$
$$(\exp X/2)\dot{X}\exp(-X/2) - (\exp(-X/2))\dot{X}\exp X/2 \quad .$$

Thus,

$$X\,\tau_X(\dot{X}) - \tau_X(\dot{X})X = e^{L_{X/2}}\cdot e^{-R_{X/2}}\cdot\dot{X} - e^{-L_{X/2}}\cdot e^{R_{X/2}}\cdot\dot{X}$$
$$= (e^{D_{X/2}} - e^{-D_{X/2}})\dot{X}$$

and we have

$$D_X(\tau_X(\dot{X})) = (e^{-D_{X/2}} - e^{D_{X/2}})\dot{X}$$

Whence the formula

$$\tau_X = \sinh(D_{X/2})/D_{X/2} \quad .$$

A comparison of the power series expansions of $(\tau_X(A))B$ and $A(\tau_X(B))$ shows

$$\mathrm{tr}[\tau_X(A))B] = \mathrm{tr}[A(\tau_X(B))]$$

if we first observe that $(D_X)^2$ restricted to $\mathscr{S}$ is self-adjoint ( the coefficients of even power terms in the expansion of sinh are all equal to zero). More specifically, we have

$$\mathrm{tr}(D_XA)B + \mathrm{tr}\,A(D_XB) = 0\ ;$$

i.e., ${}^tD_X = -D_X$ and ${}^tD_X{}^2 = D_X2$ , the transpose being with respect to the bilinear form $\mathrm{tr}\,XY$ .

Lemma 2: $\operatorname{tr}(\tau_X(Y))^2 \geq \operatorname{tr} Y^2$ for all $X, Y \in \mathcal{S}$ .

Proof: If $X$ has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ the eigenvalues of $\tau_X$ are

$$\sinh(\lambda_i/2 - \lambda_j/2)/\lambda_i/2 - \lambda_j/2 \geq 1 \quad .$$

In $P$ we introduce two metrics as follows.

Definition: Let $\rho(t)$ be a differentiable path in $P$ . The Riemanniam infinitessimal length is

$$|\dot\rho|_R{}^2 = \operatorname{tr}(\rho^{-1}\dot\rho)^2 = \operatorname{tr}(\rho^{-1/2}\dot\rho\,\rho^{-1/2})^2 \quad .$$

For any path $f$ in $P$ defined on $[a,b]$ we define Riemanniam length to be

$$\rho_R(f) = \int_a^b |\dot f|_R \, dt \quad .$$

The Euclidean infinitessimal length is

$$|\dot\rho|_{\mathcal{S}}^2 = \operatorname{tr}(\dot{\log}\,\rho)^2 \quad .$$

The Euclidean length of $f$ is defined by

$$\rho_S(f) = \int_a^b |\dot f|_{\mathcal{S}} \, dt$$

Remark: The Riemannian metric is invariant under the map $y \longrightarrow xy\,{}^tx$ , since

$$\operatorname{tr}({}^tx^{-1}\rho^{-1}x^{-1}x\,\dot\rho\,{}^tx)^2 = \operatorname{tr}(\rho^{-1}\dot\rho)^2$$

and this is the action of $GL(n,R)$ on quadratic forms by substitution.

An immediate consequence of Lemma 2 is:

*Lemma 2'*: $|\dot\varrho|_{\mathcal{S}}^2 \leq |\dot\varrho|_R{}^2$

*Proof*: We point out the formula

$$\varrho^{-1/2}\,\dot\varrho\,\varrho^{-1/2} = \tau_{\log\varrho}(\dot{\log}\,\varrho)$$

and Lemma 2 gives

$$|\dot\rho|_R{}^2 = \operatorname{tr}(\tau_{\log\rho}(\log\dot\rho))^2 \geq \operatorname{tr}(\log\varrho)^2 = |\varrho|_{\mathcal{S}}^2 .$$

*Definition*: A *geodesic* joining the points p and q of P is a path of minimal length joining p and q.

*Lemma 3*: There is a unique geodesic joining any two points of P. The geodesics issuing from the identity are the 1-parameter subgroups.

*Proof*: Since GL(n,R) is a transitive group of isometries, it suffices to consider paths joining the identity element, I, to an element p.

Let f be a path in P with f(0) = I, f(1) = p. For an $\mathcal{S}$-straight line path, $\ell$, joining I and p we get

$$\rho_{\mathcal{S}}(\ell) = \operatorname{tr}(\log p)^2 .$$

Then

$$(*)\quad \operatorname{tr}(\log p)^2 = \rho_{\mathcal{S}}(\ell) \leq \rho_{\mathcal{S}}(f) \leq \rho_R(f)$$

The 1-parameter subgroup $g(t) = \exp(t \log p)$ is a straight line in logarithmic coordinates; in particular,

$$\rho_{\mathcal{S}}(g) = \operatorname{tr}(\log p)^2 .$$

Moreover, we have $g^{-1}\cdot\dot g = \log p$, whence

$$(**)\quad \rho_R(g) = \int_0^1 \operatorname{tr}(\log p)^2 = \operatorname{tr}(\log p)^2 = \rho_{\mathcal{S}}(g)$$

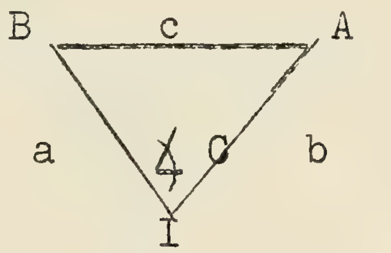
From (*) and (**) we see that $\mathrm{tr}(\log p)^2$ is the minimum length and, as claimed, the 1-parameter subgroup is a geodesic joining I to p . Since $\rho_{\mathcal{S}}(f) = \mathrm{tr}(\log p)^2$ only for $f = g$ we have the uniqueness assertion

Let ABC denote a geodesic triangle in P . Denote by a,b,c the lengths of the sides opposite the verticies A,B, and C respectively. The infinitessimal quadratic form gives a bilinear form, in the standard way, making it possible to speak of the cosine of an angle. We denote the angle at the vertex C by $\measuredangle C$ and we have the following "Law of Cosines": inequality.

Lemma 4:

$$c^2 \geq a^2+b^2 - 2ab\cos\measuredangle C$$

Proof: Since GL(n,R) operates in P as a transitive group of isometries, we can assume that the vertex C is I . Our triangle is in P



.

In logarithmic coordinates the 1-parameter subgroup joining I to B becomes a line of length a and the 1-parameter subgroup joining I to A becomes a line of length b and $\measuredangle C$ becomes an angle of the same magnitude. This is because $\exp : \mathcal{S} \longrightarrow P$ induces an isometry of the tangent space at 0 , $\mathcal{S}_0$ , to the tangent

space at $I$ , $\dot{P}_I$ . Thus we have:

$$\rho_{\mathcal{S}}(IB) = \rho_R(IB) ,$$

$$\rho_{\mathcal{S}}(IA) = \rho_R(IA) , \text{ and}$$

$$\cos_{\mathcal{S}} \measuredangle C = \cos_R \measuredangle C ,$$

where, in general, $\rho(XY)$ is the length of the path connecting $X$ and $Y$ .

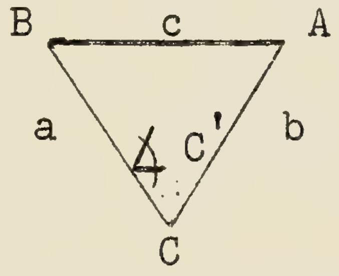
We have $\rho_{\mathcal{S}}(BA) \leq \rho_R(BA) = c$ . By the Law of Cosines (for euclidean space)we have:

$$\rho_{\mathcal{S}}(BA)^2 = a^2+b^2 - 2ab \cos \measuredangle C$$

and hence

$$\rho_R(BA)^2 = c^2 \geq a^2+b^2 - 2ab \cos_R \measuredangle C \quad .$$

Remark: It follows that the sum of the angles of a geodesic triangle does not exceed 180° . The reason is that $c \leq a+b$ assures us that there is a euclidean triangle



with $\cos \measuredangle C = \dfrac{a^2+b^2-c^2}{2ab}$ .

The inequality above yields

$$c^2 \geq a^2+b^2-2ab \cos \measuredangle C \text{ , therefore}$$

$$\cos \measuredangle C \geq \frac{a^2+b^2-c^2}{2ab} \text{ , therefore}$$

$$\measuredangle C \leq \measuredangle C' \quad .$$

Tt turns out that we get 180° as the sum of the angles only when the sides lie on 1-parameter subgroups which commute and the negative curvature arises from the fact that 1-parameter subgroups do not commute in general.

A proof of the following theorem may be found in the Memoirs paper cited above.

Theorem 1: Let $\mathcal{E}$ be a linear subspace of $\mathcal{S}$ satisfying the condition A):

$[X,[X,Y]]$ is in $\mathcal{E}$ whenever $X$ and $Y$ are in $\mathcal{E}$.

Set $\mathcal{F} = \{Y;\ Y \in \mathcal{S}$ and $\operatorname{tr} Y\mathcal{E} = 0\}$. Set $E = \exp \mathcal{E}$ and $F = \exp \mathcal{F}$.

Then the map $\mu : F \times E \longrightarrow P$ given by

$$\mu(f,e) = efe$$

is a homeomorphism.

For our purposes here we may assume condition A') --

$$E = G \cap P$$

where $G$ is an analytic group. This implies that condition A) is satisfied for if $E = G \cap P$ then $\mathcal{E} = \dot{G} \cap \mathcal{S}$. We prove here the following theorem which is apparently weaker than the one above but is actually equivalent to it:

Theorem 1': Assume $\mathcal{E}$ satisfies condition A'). Then

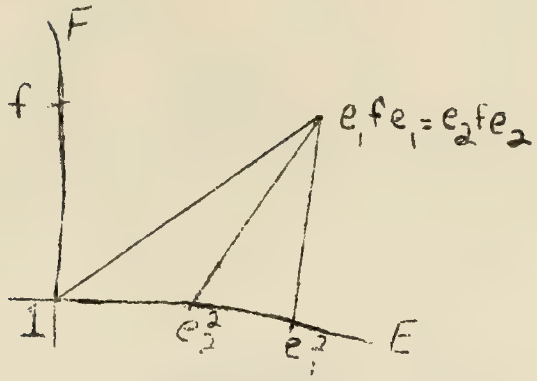
$$\mu : F \times E \longrightarrow P$$

is a homeomorphism.

Proof: First we show that $\mu$ is injective. If $x$ is an element of $E = G \cap P$ then the isometry $\sigma_x : P \longrightarrow P$ defined by

$$\sigma(y) = xyx$$

leaves $E$ invariant.



Now $e_1fe_1 = \sigma_{e_1}(f)$ and, since $F$ is orthogonal to $E$, the triangle $(1)(e_1^2)(efe)$ is a right triangle. Given $e_1f_1 = e_2f_2$ with $e_i \in E$ for $i = 1,2$ and $f \in F$ we have the geodesic triangle $(e_1^2)(e_2^2)(e_1fe_1)$ the sum of whose angles total more than 180° unless $e_1^2 = e_2^2$; i.e., $e_1 = e_2$ and $\mu$ is injective.

Since the cosine of a right angle is equal to 0 we have

$$|efe|^2 \geq |e|_R^2 + |f|_R^2 .$$

Hence, if either $|e|_R^2$ or $|f|_R^2$ go to infinity so does the left hand side of the above inequality and we see that the inverse image (under $\mu$ ) of a bounded set is bounded. As the inverse image of a closed set is closed ( $\mu$ is continuous) we see that the inverse image of a compact set is compact. Thus, the image $\mu(F \times E)$ is closed in $P$. The principle of invariance of domain tells us that if we have a continuous, one-to-one mapping of a manifold $M$ into a manifold $N$, then the image is open. Hence $\mu(F \times E)$ is open in $P$ and as $P$ is connected $\mu(F \times E) = P$ and the continuous,

one-to-one mapping $\mu$ is a homeomorphism.

Theorem 2: Let $\mathcal{E}$ satisfy condition A') . Then the map

$$\nu : O(n,R) \times F \times E \longrightarrow GL(n,R) \text{ given by}$$

$$\nu(k,f,e) = kfe$$

is a homeomorphism.

Proof: First, $\nu$ is injective -- for if $k_1f_1e_1 = k_2f_2e_2$ then

$$e_1f_1{}^tk_1k_1f_1e_1 = e_2f_2{}^tk_2k_2f_2e_2$$

and since ${}^tk_1k_1 = {}^tk_2k_2 =$ the identity we have

$$e_1f_1{}^2e_1 = e_2f_2{}^2e_2$$

and the univalence of $\mu$ gives $e_1 = e_2$ , $f_1{}^2 = f_2{}^2$ and hence $f_1 = f_2$ .

Given $g \in GL(n,R)$ we want to solve the equation $g = kfe$ with $e \in E$ , $f \in F$, and $k \in O(n,R)$ . If that has been done we find $ef^2e = {}^tgg$ is positive definite and symmetric; i.e., $ef^2e \in P$ . Since $\mu$ is surjective we can find $e,f^2$ and hence $f$ such that

$$ef^2e = {}^tgg ;$$

i.e., ${}^tg^{-1}ef(fe) = G$ . This implies

$$feg^{-1} = {}^t({}^tg^{-1}ef) = ({}^tg^{-1}ef)^{-1}$$

and so $feg^{-1} \in O(n,R)$ and we find that $\nu$ is surjective. Hence $\nu$ is a homeomorphism.

Let H be a closed (connected) subgroup of GL(n,R) , such that $H \cap P = E$ and $H = (H \cap O(n,R))\cdot(H \cap P)$ . Let $H \cap O(n,R)$ operate in $O(n,R)$ on the right by right translations and in F

on the left by inner automorphism; i.e., for $x \in H \cap O(n,R)$ and $y \in F$ we may write $y = \exp Y$ with $Y \in \mathcal{F}$ and then $xyx^{-1} = \exp (xYx^{-1})$ and $xYx^{-1}$ is perpendicular to $x \mathcal{E} x^{-1} = \mathcal{E}$ so that $xyx^{-1} \in F$. Then the above result may be interpreted as follows:

Corollary: With the above conditions on $H$ we have

$$GL(n,R)/_{H} = K \underset{H \cap K}{\times} F$$

where $K = O(n,R)$.

The space $K \underset{H \cap K}{\times} F$ is a fiber-bundle over ${}^{K}/_{H \cap K}$ (which is compact) with fiber $F$ (which is euclidean).

Proposition: Let $G$ and $G'$ be closed, connected subgroups of $GL(N,R)$ with

$$G = (G \cap O)\cdot(G \cap P)$$
$$G' = (G' \cap O)\cdot(G' \cap P)$$

as topological direct products. Then, as a topological direct product $G = (G \cap O)\cdot(G \cap F)\cdot(G' \cap P)$ where $G \cap F = \exp(\mathcal{F} \cap \dot{G})$ and $\mathcal{F}$ is the orthogonal complement to $(\dot{G}' \cap \mathcal{S})$ in $\mathcal{S}$. Moreover, $G \cap P = \exp(\dot{G} \cap \mathcal{S})$.

Proof: We first prove the assertion about $G \cap P$. Since $\exp(\dot{G} \cap \mathcal{S})$ is closed in $G \cap P$ we know that $(G \cap O)\cdot\exp(G \cap \mathcal{S})$ is closed in $G$. It is a submanifold of the same dimension and hence is open in $G$. Hence,

$$G = (G \cap O)\cdot\exp(\dot{G} \cap \mathcal{S})$$

and thus we see that $G \cap P = \exp(\dot{G} \cap \mathcal{S})$.

Now, for any $g \varepsilon G$ we can write $g = kfe$ with $k \varepsilon O$, $f \varepsilon F$, and $e \varepsilon G' \cap P$. Then

$$ef^2e = {}^t gg \;\varepsilon\; G$$

and $f^2 \varepsilon G \cap P$, so $f \varepsilon \exp(\dot{G} \cap \mathcal{S}) \cap \exp \mathcal{F} = \exp(\dot{G} \cap \mathcal{F})$. Thus

$$G = (G \cap O)(G \cap F)(G' \cap P)$$

and $G \cap F = \exp(\dot{G} \cap \mathcal{F})$.

We now take up the conjugacy of maximal compact subgroups.

Theorem: Let $G$ be a closed (notnecessarily connected) subgroup of $GL(n,R)$ such that

$$G = (G \cap O)\cdot(G \cap P)$$

with $G \cap P = \exp(\acute{G} \cap \mathcal{S})$ .

Then any compact subgroup of $G$ is conjugate to a subgroup of $G \cap O$ by means of an inner automorphism of $G$ .

Proof: Let $K$ be a compact subgroup of $G$ . Consider the operation of $G$ on $G \cap P$ as follows:

$$\sigma_x : \rho \longrightarrow x\,\rho\,{}^t x$$

for $x \in G$ and $\rho \in G \cap P$ . Setting $C = G \cap O$ we define the map $\varphi : G/C \longrightarrow G \cap P$ by $\varphi(gC) = g\,{}^t g$ . $G/C$ is a G-space by means of the operation

$$\lambda_x : gC \longrightarrow xgC ,$$

for $x \in G$ . The equation

$$\varphi(\lambda_x gC) = \varphi(xgC) = xg\,{}^t g\,{}^t x = \tau_x\,\varphi(gC)$$

shows that $\varphi$ gives an equivalence of $G/C$ and $G \cap P$ as G-spaces. Also $Kx_0C = x_0C$ if and only if $x_0^{-1}Kx_0 \subset C$ . Thus to prove the conjugacy it is enough to prove that $K$ admits a fixed point in $G \cap P$ . The idea will be to make a comparison with a compact group operating in euclidean space.

We introduce the function $F$ on $G \cap P$ --

$$F(\rho) = \int_K \rho_R(\rho, k(a))^2\,dk$$

where $a$ is a fixed point in $G \cap P$ (this is the analogue of the centroid in the euclidean case). Here we are

writing $k(a)$ for $ka^tk$ . $F(\rho)$ is continuous and non-negative and hence has a minimum value. Choose $\rho_0$ so as to minimize $F$ . As the map $a \longrightarrow \rho_0^{-1/2}\, a\, \rho_0^{-1/2}$ is an isometry, we may replace $a$ by $\rho_0^{-1/2} a\, \rho_0^{-1/2}$ and find that $I$ minimizes $F$ .

By Rolle's theorem the directional derivatives of $F$ as $I$ are then zero for all direction. On the other hand, we can differentiate under the integral sign getting

$$\delta F(I) = \underset{K}{\phantom{x}} \delta\rho_R(I,k(a))^2 dk$$

Now, for $p$ and $q$ in $G \cap P$ we have

$$\begin{aligned}
\rho_R(p,q)^2 &= \rho_R(I,p^{-1/2}qp^{1/2})^2 \\
&= \rho\mathcal{S}(I,\ p^{-1/2}qp^{1/2})^2 \\
&= \mathrm{tr}[\log(p^{-1/2}qp^{1/2})]^2 \\
&= \mathrm{tr}\ p^{1/2}[\log(p^{-1/2}qp^{-1/2})]^2p^{-1/2} \\
&= \mathrm{tr}[\log(pq^{-1})]^2
\end{aligned}$$

as we see by looking at the power series expansion for log. In general, given a power series $f(x) = \sum_n a_n x^n$ we find

$$\delta\ \mathrm{tr}\ f(p) = \frac{d}{dt}\Big|_{t=0} \mathrm{tr}\ f(p(t)) = \frac{d}{dt}\Big|_{t=0} \sum_n \mathrm{tr}(a_n p(t)^n) \ .$$

Setting $\dot{p}(0) = \delta p$ and observing that $\frac{d}{dt}(p\cdot\dot{p}) = \dot{p}\cdot p + p\cdot\dot{p}$ we have:

$$\delta \operatorname{tr} f(p) = \sum_{n} \sum_{r+s=n-1} \operatorname{tr} a_n p^r(\delta p)p^s$$

$$= \sum_{n} \operatorname{tr} n\, a_n p^{n-1} \delta p$$

$$= \operatorname{tr} f(p)\delta p \ .$$

Holding $q$ constant we have

$$\delta \operatorname{tr} \log(pq^{-1})^2 = \operatorname{tr}(2 \log(pq^{-1})\cdot(pq^{-1})^{-1}\cdot q^{-1}\delta p) \ ,$$

whence the equation

$$\text{(I)} \quad 0 = \delta F(I) = \int_K \operatorname{tr}(2 \log(q^{-1})\delta p)dk$$

$$= -2 \int_K \operatorname{tr}(\log k(a)\cdot \delta p)dk$$

for all $\delta p$ in $\dot{G} \cap \mathcal{S}$ . Thus $\int_K \operatorname{tr} \log k(a)dk = 0$ .

We then compare this data with that we get for the minimum of the function

$$L(\varrho) = \int_K \rho_{\mathcal{S}}(\varrho, k(a))^2 dk$$

on the space $G \cap P$ . We have

$$L(\varrho) = \int_K \operatorname{tr}[\log \varrho - \log k(a)]^2 dk$$

and the equation

$$\text{(II)} \quad 0 = \delta L = \int_K 2\operatorname{tr}(\log \varrho - \log k(a))\delta(\log \varrho)dk$$

for all $\delta(\log \varrho)$ in $\dot{G} \cap \mathcal{S}$. For a point $\varrho$ which minimizes $L$ we have:

$$\int_K 2(\log \rho - \log k(a))dk = 0$$

and the solution for $\log \rho$ is unique; viz.,

$$\log \rho = \frac{\int_K [\log k(a)]dk}{2\int_K dk} = 0 \ .$$

Since $\rho$ is unique and $\rho = I$ satisfies condition (II) we see that $I$ minimizes $L(\rho)$ and for $p \neq I$ and $p \in G \cap P$

$$\begin{aligned} F(I) = L(I) < L(p) &= \int_K \rho_{\mathcal{S}}(p,k(a))^2 dk \\ &\leq \int_K \rho_R(p,k(a))^2 dk \\ &= F(p) \end{aligned}$$

Thus we find that $I$ is the unique minimum for $F$. Now $K$ operates as a group of isometries in the Riemanniam metric; in particular, $F(k(I)) = F(I)$ and so $k(I) = I$. It follows that $K$ has a fixed point in $G/G \cap O$ and hence $K$ is conjugate to a subgroup of $G \cap O$.

Note: We see in this way that $G \cap O$ is a maximal compact subgroup of $G$. This assertion follows also from the fact that $G/G \cap O$ is euclidean space and that euclidean space admits no non-trivial fibration with compact fibers.

# LECTURE XLI

**Proposition** Let $\mathcal{G}$ be a real, semi-simple Lie algebra and $\mathcal{G}'$ a semi-simple subalgebra of $\mathcal{G}$. Identifying the complexification $\mathcal{G}_c'$ of $\mathcal{G}'$ with a subalgebra of the complexification $\mathcal{G}_C$ of $\mathcal{G}$ in the obvious manner, we let $\mathcal{G}_K$ and $\mathcal{G}_K'$ be invariant compact real forms of $\mathcal{G}_C$ and $\mathcal{G}_C'$ respectively.

Then there exists an inner-automorphism r of $\mathcal{G}$ sending $\mathcal{G}_K'$ into $\mathcal{G}_K$.

Proof Let $\tau$ be the conjugation of $\mathcal{G}_C$ with respect to $\mathcal{G}_K$. Let $G_C$ denote the automorphism group of $\mathcal{G}_C$ (recall that $G_C$ is an algebraic group). The analytic subgroup determined by $\mathcal{G}_K'$ is compact. Hence, by the theorem of Lecture XL there exists an automorphism p in $G_C$ sending $\mathcal{G}_K'$ into $\mathcal{G}_K$. Now $\mathcal{G}_C$ is a semi-simple real Lie algebra and

$$\mathcal{G}_c = \mathcal{G}_K + \sqrt{-1}\,\mathcal{G}_K$$

is a Cartan decomposition for $\mathcal{G}_C$. Then $G_C = G_K \cdot \exp\sqrt{-1}\,\mathrm{ad}\,\mathcal{G}_K$ since $G_C$ is algebraic over the reals and $G_K = \mathcal{G}_K$. Writing p as an element $g_K \cdot \exp\sqrt{-1}\,\mathrm{ad}\,g_K'$ and observing $G_K(\mathcal{G}_K) \subset \mathcal{G}_K$ we see that we can assume $p \in \exp\sqrt{-1}\,\mathrm{ad}\,\mathcal{G}_K$ and

$$p(\mathcal{G}_K') \subset \mathcal{G}_K$$
$$\bar{p}(\mathcal{G}_K') \subset \mathcal{G}_K .$$

We write $\bar{p}p^{-1} = k \cdot q$ with $k \in G_k$ and $q \in \exp\sqrt{-1}\,\mathrm{ad}\,\mathcal{G}_K$. For $x \in p(\mathcal{G}_K')$ we have $k \cdot q(x) = \bar{p}p^{-1}(x) \in \bar{p}(\mathcal{G}_K') \subset \mathcal{G}_K$. So the element $x' = q(x)$ is in $\mathcal{G}_K$ and

$$x' = \tau(q(x)) = \tau q \tau^{-1}(\tau(x)) .$$

Because $q \in \exp \sqrt{-1}\, \mathrm{ad}\, \mathfrak{g}_K$ we have $\tau q \tau^{-1} = q^{-1}$ and hence $x' = q^{-1}(x)$; i. e.

$$q(x) = q^{-1}(x) \quad \text{and then}$$

$$q^2(x) = x$$

for all $x \in p(\mathfrak{g}_K')$. Since the subgroup of $G_C$ keeping $x$ fixed is algebraic and contains $q^2$ it contains $q^t$ for all real $t$ ($q$ is a semi-simple automorphism with positive eigenvalues); i. e., $q^t(x) = x$ for all real $t$.

We set $r = (p(p^{-1}p^{-2}p^{-1})^{1/2}p)^{1/2}$ . We recall that

$$\bar{p}p^{-1} = k \cdot q \ .$$

Then

$$\tau(p^{-1}\bar{p})^{-1} = \tau(kq)^{-1}$$

$$p^{-1}\bar{p} = q \cdot k^{-1}, \quad \text{and so}$$

$$p^{-1}p^{-2}p^{-1} = q^2 \ .$$

If $y \in \mathfrak{g}_K'$ then

$$p(p^{-1}p^{-2}p^{-1})^{1/2}p(y) =$$

$$pqp(y) = p^2(y); \text{ i. e. } r^2(y) = p^2(y) \ .$$

We claim $r\mathfrak{g}_K' \subset \mathfrak{g}_K$ . To see this let $y \in \mathfrak{g}_K'$ . We have

$$\tau(r(y)) = \tau r \tau^{-1}(\tau(y)) = r^{-1}(\tau(y)).$$

Since $p(y) \in \mathfrak{g}_K$ we have $\tau p(y) = p(y)$ - hence,

$$p^{-1}\tau(y) = p(y)$$

$$\tau(y) = p^2(y) \ .$$

Now,

$$r^{-1}(\tau(y)) = r^{-1}p^2(y) = r^{-1}r^2(y) = r(y)$$

and we have

$$\tau(r(y)) = r(y) .$$

We may write $r = \exp \sqrt{-1}\ \mathrm{ad}\ X$ and since $\bar{r} = r$ we have $X \in \mathrm{ad}\ (\mathfrak{g} \cap \sqrt{-1}\,\mathfrak{g}_K)$ .

Definition Let $G_C$ be a Lie group whose Lie algebra $\dot{G}_C$ has the structure of a complex Lie algebra compatible with its canonical real Lie algebra structure. By a compact real form of $G_C$ we mean a compact subgroup $G_K$ satisfying

1) $\dot{G}_K$ is a real form of $\dot{G}_C$; i. e. $\dot{G}_C = \dot{G}_K \otimes_R C$

2) $G_C$ is generated by $G_K$ and the connected component of the identity on $G_C$.

If, moreover, $G_C \subset GL(n,C)$ and $G_K$ is stable under complex conjugation we call $G_K$ an invariant compact real form of $G_C \cap GL(n,R)$

Theorem Let $G$ and $G'$ be algebraic subgroups of $GL(n,R)$. Assume $G$ and $G'$ are self-adjoint with respect to the positive-definite bilinear forms $B$ and $B'$ respectively. Thenthere is an element $r$ in $G$ such that both $G$ and $G'$ are self-adjoint with respect to the transform of $B$ under $r$.

Proof Let $\mathcal{O}, \mathcal{S}$ , and $P$ denote respectively the set of orthogonal, symmetric, and positive-definite symmetric elements with respect to $B$. Then,

$$G = (G \cap O)\cdot(G \cap P)$$

and $G \cap P = \exp(\dot{G} \cap \mathcal{S})$. Upon extending the ground field to $C$ we find for the complexification

$$G_C = (G_C \cap U)\cdot(G_C \cap P_C)$$

where $U$ is the unitary group on $C^n$ of the hermitian from $B_C$ and $P_C$ is the set of positive-definite hermitian elements. Also, we have $(G_C \cap P_C) = \exp(\dot{G}_C \cap \sqrt{-1}\dot{U})$

$G_C \cap U$ is an invariant compact real form of $G_C$. Similary $G_C'$ has a compact real from $G_C' \cap U'$. Applying the proof of the preceding proposition we obtain the desired element $r$.

Remark 1. We can also make a finite chain of such groups simultaneously self-adjoint.

Remark 2. An algebraic linear group is self-adjoint with respect to some form if and only if it is fully reducible. (cf. Mostow self-adjoint groups. Ann. of Math. 195)

We have the following corollary to the Proposition above.

Corollary Given $\mathcal{G}' \supset \mathcal{G}$, both semi-simple real Lie algebras there exist compatible Cartan decompositions; i. e.,

$$\mathcal{G} = K + \mathcal{E}$$

$$\mathcal{G}' = K' + \mathcal{E}'$$

with $K \supset K'$ and $\mathcal{E} \supset \mathcal{E}'$; viz., take invariant compact real forms $\mathcal{G}_K'$ and $\mathcal{G}_K$ with $\mathcal{G}_K' \subset \mathcal{G}_K$. Take $K = \mathcal{G} \cap \mathcal{G}_K$, $\mathcal{E} = \mathcal{G} \cap \sqrt{-1}\,\mathcal{G}_K$, $K' = \mathcal{G}' \cap \mathcal{G}_K$ and $\mathcal{E}' = \mathcal{G}' \cap \sqrt{-1}\,\mathcal{G}_K$.

# LECTURE XLII

We denote these last lectures to a discussion of "convariant fiberings" of manifolds on which a Lie group operates transitively.

**Definition** Let $S$ be a manifold on which the Lie group $G$ operates transitively. A covariant fibering of $S$ is a differentiable fibering into euclidean fibers such that:

1) The fibers are permuted transitively by some maximal compact subgroup $M$ of $G$,

2) The subgroup $M^F$ of $M$ keeping any fiber $F$ invariant operates as a linear group on $F$ (and in particular keeps a point $x_F$ of $F$ fixed since it is compact).

Thus $S = M \underset{M^F}{\times} F$

Denote by $G^y$ the isotropy subgroup of a point $y \in S$. Let $G' = G^x$ where $x$ is a fixed point in $F$ with $M^F x = x$. Then we have $M^F = M \cap G'$ and $S = G/G'$ so that

$$G/G' = M \underset{M \cap G'}{\times} F .$$

We can see that $M \cap G'$ is a maximal compact subgroup of $G'$ as follows - by Corollary 3 below, the maximal compact subgroups of an analytic group are conjugate under inner automorphism. Consequently, $G' \cap g^{-1}Mg$ is a maximal compact subgroup of $G'$ for some $g \in G$. Then $gG'g^{-1} \cap M$ is maximal compact in $gG'g^{-1} = G^{gx}$. Let $F'$ be the fiber through the point $gx$.

Then $M^{F'} \supset M \cap G^{gx} \approx M^F$. But $M^F$, being conjugate to $M^F$ by an inner automorphism of $M$, has the same dimension and the same number of connected components as $M^F$. Hence $M^{F'} = M \cap G^{gx}$, and $M^F$ is isomorphic to a maximal compact subgroup of $G' \approx G^{gx}$. Since all the maximal compact subgroups of $G'$ are conjugate under inner automorphism, we conclude that $M^F$ is maximal compact in $G'$.

The uniqueness of covariant fiberings is proved in Mostow, Covariant Fiberings of Klein Spaces, Amer. J. of Math, v. 77, 1955 (we refer to this paper as CFI). The existence of covariant fiberings is based on special decompositions of $G$ and $G'$. We outline, below, a proof of the fact that any homogeneous space $G/G'$ with $G'$ a closed analytic subgroup of the analytic group $G$ admits a covariant fibering.

Lemma 1. Let $G$ be a locally compact group such that

$$(1) \longrightarrow V \longrightarrow G \longrightarrow K \longrightarrow (1)$$

is exact, where $V$ is a normal vector subgroup and $K$ is a compact group. Then the extension splits; i. e., $G = K \cdot V$ (semi-direct). Moreover, any compact subgroup of $G$ is conjugate to a subgroup of $K$ by an inner automorphism.

Proof Let $\pi : G \longrightarrow G/V$ be the natural projection. Let $\varphi$ be a map of $G/V \longrightarrow G$ which is piecewise continuous [on a finite family of Borel subsets of $G$(note - we are using the Haar measure in $G$ here)], and such that $\pi \circ \varphi$ = identity~~. We define~~

a "factor set function" $f: G/V \times G/V \longrightarrow V$ by the formula

$$f(x,y)\varphi(xy) = \varphi(x)\varphi(y)$$

for $x,y \in G/V$. The associate identity gives

$$\varphi((xy)z) = \varphi(x(yz))$$

whence

$$f(xy,z)\varphi(xy)\varphi(z) = f(x,yz)\varphi(x)\varphi(yz)$$

and then

$$\begin{aligned} &f(xy,z)f(xy)\varphi(x)\varphi(yz) = \\ &f(x,yz)x[f(y,z)]\varphi(x)\varphi(y)\varphi(z) \end{aligned}$$

where $x[v] = \varphi(x)v\varphi(x^{-1})$ for $x \in G/V$ and $v \in V$. We use additive notation in $V$ and write

$$\text{(I)} \quad x[f(y,z)] - f(xy,z) + f(x,yz) - f(x,y) = 0 .$$

In the context of cohomology groups, this says that $f$ is a 2-cocylce in the non-homogeneous standard oomplex. We normalize the Haar measure in $G$ so that the total measure of $G/V$ is 1. Integrating (I) over $G/V$ with respect to $z$ we set

$$g(x) = \int_{G/V} f(x,z)dz$$

Then

$$g(xy) + f(x,y) = g(x) + x[g(y)], \quad \text{or}$$

$$f(x,y) = xg(y) - g(xy) + g(x); \text{ i. e.}$$

f is a coboundary.

$$\begin{aligned} \text{Set}\quad \theta(x) &= g(x)\varphi(x). \quad \text{Then} \\ \theta(x)\theta(y) &= g(x)^{-1}\varphi(x)g(y)^{-1}\varphi(y) \\ &= g(x)^{-1}x[g(y)]^{-1}\varphi(x)\,\varphi(y) \\ &= g(x)^{-1}x[g(y)]^{-1}f(x,y)\,\varphi(xy) \\ &= g(xy)^{-1}\varphi(xy) = \theta(xy)\ . \end{aligned}$$

Thus $\theta$ is a homomorphism and, being the integral of a measurable function, is continuous. Thus, the image $K = \theta(G/V)$ is compact and $G = K\cdot V$ (semi-direct) .

For the second part of the assertion we argue as follows. Consider the canonical operation of $G$ on $G/K$. Since $G = V\cdot K$ (semi-direct), we may identify the operation on $G$ or $G/K$ with affine transformations on $V$. $K'$ being a compact group of affine transformations on $G/K$, keeps some point fixed -- indeed the centroid of any orbit. Thus

$K'v_0K = v_0K$ for some $v_0 \in V$ and $v_0^{-1}K'v_0 \subset K$ .

Note: In the special case that $K'V = G$, the forgoing proof may be interpreted as the assertion $H^1(G/V, V) = 0$. Namely let $\lambda: G/V \longrightarrow K'$ be a homomorphism with $\lambda \circ \pi = \mathrm{id}$. Then $\lambda$ determines a one cochain $c(x)$ via: $c(x)\theta(x) = \lambda(x)$, and conversely every one cochain determines such a $\lambda$. We have

$c(xy) = c(x) + x[c(y)]$.

and $v_0K'v_0^{-1} \subset K \iff$

$$c(x) = v_0 - x[v_0]\ .$$

This states that $c(x)$ is a coboundary .

Corollary 1. The conclusion of Lemma 1 is valid if $V$ is a solvable closed normal analytic simply - connected subgroup.

Proof This may be shown by induction on the length of the sequence of subgroups

$$V = V_0 \supset V_1 \supset \cdots \supset V_n = (1)$$

where $V_{i-1} = \overline{[V_i, V_i]}$.

Corollary 2. Let $G$ be a locally compact group and $R$ a closed normal solvable analytic subgroup such that $G/R$ is compact. Then there is a compact subgroup $K$ in $G$ such that $G = KR$ (not necessarily semi-direct). Moreover, any compact subgroup of $G$ is conjugate to $K$ via an inner automorphism.

Corollary 3. Let $G$ be an analytic group and let $M$ be a maximal compact analytic subgroup. Then any compact subgroup of $G$ is conjugate to a subgroup of $M$ by an inner automorphism.

Proof. Let $R$ be the radical of $G$. If $\dim R = 0$, then $G$ is semi-simple and the result is known for the adjoint group of $G$. There upon one deduces the result for $G$ itself. If $\dim R > 0$ the result is already proved mod $R$ and we apply Corollary 2.

Remark A maximal compact analytic subgroup $M$ of any analytic group $G$ is a maximal compact subgroup; for, if $M \subset P$ and $P$ is a compact subgroup, then $P$ is conjugate to a subgroup $P'$ of $M$ and we have

$$M' \subset P' \subset M \subset P$$

with $M' \approx M$ and $P' \approx P$. Since $\dim M' = \dim M$, $M'$ is open as well as closed in $M$. Thus $M' = P' = M$ and $M = P$.

Exercise (Levi Decomposition) Let $G$ be an analytic group and $R$ its radical. Prove that $G$ admits a semi-simple analytic subgroup $L$ such that $G = LR$. Moreover all such $L$ are conjugate by an inner automorphism.

Hint:

Let $\mathcal{G}_{\mathbb{C}} = \dot{G} \underset{R}{\otimes} \mathbb{C}$ and let $Z_2$ operate on $\mathcal{G}_{\mathbb{C}}$ via complex conjugation. Let $G_{\mathbb{C}}$ be a simply connected analytic group with Lie algebra $G_{\mathbb{C}}$, and let $Z_2$ operate in $G_{\mathbb{C}}$ via complex conjugation on $\mathcal{G}_{\mathbb{C}}$. Form the semi-direct products $Z_2 \ltimes G_{\mathbb{C}}$ and $Z_2 \ltimes (G_{\mathbb{C}}/R_{\mathbb{C}})$ with $R_{\mathbb{C}}$ the radical of $G_{\mathbb{C}}$.

1) Prove that any maximal compact subgroup of $Z_2 \ltimes G_{\mathbb{C}}/R_{\mathbb{C}}$ is a compact real form if it contains $Z_2$

2) Let $L_K$ be a subgroup of $Z_2 \ltimes G_{\mathbb{C}}$ which contains $Z_2$ and which projects onto a compact real form of $Z_2 \ltimes G_{\mathbb{C}}/R_{\mathbb{C}}$. Let $\mathcal{L}_{\mathbb{C}} = \dot{L}_K \underset{R}{\otimes} \mathbb{C}$. Then
$$\dot{G} = (\dot{G} \cap \mathcal{L}_{\mathbb{C}}) + \dot{R} \quad \text{(semi-direct)}$$

3) Prove that any semi-simple analytic subgroup of $G$ is conjugate to a subgroup of $L$ by an inner automorphism.

Definition: Let $G$ be an analytic group and let $\Gamma$ be a group of automorphisms of $G$. By a $\Gamma$-invariant exp-set $S$ in $G$ we mean a subset $S$ of $G$ for which there is a family of independent linear subspaces $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_r$ in $G$, each invariant under $\Gamma$ ($\Gamma$ induces a group of automorphisms in $G$, cf. Lecture XII), such that the map

$$s_1 + s_2 + \ldots + s_r \longrightarrow \exp s_1 \exp s_2 \ldots \exp s_r .$$

is a homeomorphism of $\mathcal{S}_1 + \mathcal{S}_2 + \ldots + \mathcal{S}_r$ onto $\mathcal{S}$.

If $M'$ is a subgroup of $G$, we shall denote by $\Gamma_{M'}$, the group of inner automorphisms $g \longrightarrow m^{-1} g m$ with $m \in M'$.

Theorem: Let $G$ be an analytic group and let $G'$ be a proper closed analytic subgroup. Let $M'$ be a maximal compact analytic subgroup of $G'$ and let $M$ be a maximal compact analytic subgroup of $G$ with $M' \subset M$. Then there exist $\Gamma_{M'}$ invariant exp-sets $F$ and $E'$ such that

$$G = M \cdot F \cdot E'$$

$$G' = M' \cdot E'$$

as topological direct products.

The proof comes after a series of Lemmas.

Lemma 1: If the Theorem is true for $G$, it is true for a covering group of $G$.

The proof of Lemma 1 is left as an exercise.

Lemma 2: If the Theorem is true whenever $G'$ is a maximal proper analytic subgroup of $G$, then it is true for any $G'$.

This can be proven directly [cf. Mostow, Covariant Fiberings of Klein Spaces II, Amer. J. Math., v.84, p.468 (hereafter referred to as CF II)].

Lemma 3: Let $G$ be an analytic semi-simple linear group and let $G'$ be a maximal analytic subgroup. Then either $G'$ is the connected component of the identity of an algebraic, self-adjoint group or $G = KG'$ with $K$ a compact subgroup.

See Mostow, On Maximal Subgroups of Real Lie Groups, Annals of Math., v.74, 1961, Theorem 3.2 and 1.5 . See also, Mostow, Self-adjoint Groups, Ann. of Math., v.62, 1955, Theorems 5.1 and 7.1 .

We now formally begin the proof of the Theorem by induction on dim $G$.

Lemma 4: The Theorem is true if $G = KG'$ with $K$ a compact subgroup.

Proof: By induction we have

$$G' = M' \cdot E'$$

with $M'$ a maximal compact analytic subgroup of $G$. The subgroup $M'$ is conjugate to a subgroup of $K$. After changing $K$ via an inner automorphism, we may assume that $K \cap G' = M'$; then

$$G = KG' = KM'E' = KE' .$$

The mapping of the manifolds $\varphi: K \times E \longrightarrow G$ given by

$\varphi(k,e) = ke$ is continuous and bijective and is, thus, a homeomorphism.

Lemma 5: The Theorem is true if either

a) $G'$ contains a non-trivial abelian analytic subgroup normal in $G$, or

b) $G = G' \times W$ (semi-direct) with $W$ a vector subgroup normal in $G$.

The proof by induction in each case is immediate.

Lemma 6: The Theorem is true if $G = G'W$ with $W$ a non-trivial abelian analytic subgroup normal in $G$.

Proof: The connected component $(G' \cap W)_1$ is normal in $G$. By Lemma 5 a), we can assume it to be trivial and thus we can assume that $G' \cap W$ is discrete. Let $T$ be the maximum toroid in $W$. $T$, being stable under inner automorphism of $G$, is a normal subgroup, and $G'T$ is a closed subgroup of $G$. Thus $G = G'T$ or $T \subset G'$. In the former case, Lemma 4 gives the result. In the latter case, we have $T = (1)$ and $W$ is a vector group. This is the case that remains to be discussed.

Let $U$ denote the minimum analytic subgroup of $W$ containing $G \cap W$. Then $U/G \cap W$ is compact so that $G'U$ is closed in $G$. Thus either $U = (1)$ and $G = G' \times U$ (semi-direct) or $G = G'U$. In the former case we are done. Assume the latter case. Now $G' \cap U$ being central in $G$, the inner automorphisms of $G$ must leave the elements of $U$ fixed. Thus $U$ is central in $G$.

Let $M$ be a maximal compact subgroup of $G$. By the induction hypothesis choose a decomposition $\bar{M}\cdot\bar{E} = G/U$. By Lemma 1, we may assume that $MU/U = \bar{M}$. Thus

$$ {}^{G/M}\Big/_{MU/M} = G/MU = {}^{G/U}\Big/_{MU/U} = \bar{M}\bar{E}/\bar{M} = \bar{E} . $$

Moreover, $MU/U = U/M\cap U = U$, since the compact subgroup $M\cap U$ is trivial. Consequently, $G/M$, the total space of a fiber bundle with euclidean fiber and euclidean base, is simply-connected. Then the quotient space $G/G'M = {}^{G/M}\Big/_{G'M/M}$, being the image of $G/M$ under a fiber map with connected fibers, is simply-connected. But $G'$ is normal in $G = G'U$ and the quotient group $G/G'M = U/U\cap G'M$ is a compact, abelian, analytic group; since it is simply-connected it consists of the identity alone and $G = G'M$. The Theorem now follows from Lemma 3.

Lemma 7: Let $G$ be an analytic group with radical $R$. Let $G'$ be a closed analytic subgroup of $G$. Assume that for every non-trivial analytic normal subgroup $V$ of $G$, the topological closure of $G'V$ is $G$. Then there is an analytic normal subgroup $W$ of $G$ with $G = G'W$.

This is Proposition 2.1 of CF I.

The proof of the Theorem now runs as follows. By Lemma 2, we may assume that $G'$ is a maximal closed analytic proper subgroup of $G$. If $G$ has no radical and $G$ is linear, then either $G'$ is self-adjoint or $G = G'K$ with $K$ compact. In the former case, the result has been proved. In the latter case, apply Lemma 4. If $G$ has a radical, then we can assume that

$a'$ has no analytic, abelian non-trivial subgroup normal in $G$. Then for any abelian analytic non-trivial subgroup $V$ we have $\overline{G'V} = G$. By Lemma 7, $G = G'W$ with $W$ a non-trivial analytic abelian subgroup. The result now follows by Lemma 6.

Corollary: A homogeneous space with trivial homotopy groups is homeomorphic to euclidean space.

Proof: $\pi_k(G/G') = \pi_k(M/M \cap G') = 0$ for all $k$ and hence $M/M'$ is a point. Thus

$$G/G' = M \underset{M' \cap G}{\times} F = F$$

Remark: The special case that $G' = M$, a maximal compact analytic subgroup of $G$, is noteworthy. Then we find

$$G = M \cdot E$$

with $E$ a euclidean subspace, invariant under the inner automorphisms by elements of $M$.

Remark 2: The existence of a covariant fibering is proved in CF II for the somewhat more general case that $G$ and $G'$ have a finite number of connected components. Thus, if $G$ and $G'$ have a finite number of connected components, $G/G'$ can be deformed by retraction to $M/M'$, $M$ being a maximal compact subgroup of $G$.

Exercise: Let $M$ be a maximal compact subgroup of the analytic group $G$. Let $S$ be a subset of $M$, and let $\gamma$ be an inner automorphism of $G$ with $\gamma(S) \subset M$. Then there is an element $k \in M$ such that $\gamma$ coincides on $S$ with the inner automorphism by $k$.

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